Simply Subtracting Squares

Lucas Jacobs and Joshua Wolford

Sometimes, when an elementary school teacher wants to help their students practice subtraction (or they don't particularly feel like teaching) they will make their students play a game involving diffy boxes, squares with an integer on each of their corners. The rule to go from one box to the next is to place the difference between each pair of adjacent corners in the middle of the edge connecting them.

Explicitly, if a and b are the values of the two corners, then the number $|a-b|$is placed on the edge. The numbers on the edges now form a new box, so this process can be applied again and again until the boxes get too small to keep drawing.

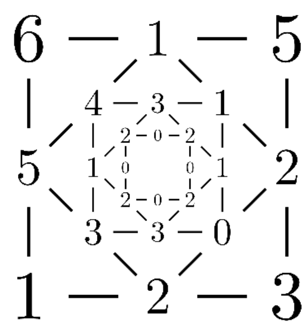

Figure 1: A diffy box.

If we use an imaginary pen that can draw infinitely thin lines, we can continue this process indefinitely. In *Figure 1*, the process has been continued until the box with a zero at each corner has been reached. From here, nothing interesting happens because each new box we draw will also be this box with only zeros. At this point, the students drawing these boxes are sure to grow bored and leave their desks to pursue more interesting uses of their time. This poses a problem for teachers - is there a box they can give their students that will never turn boring by eventually turning into the box with a zero at each corner? The reader may wish to attempt this problem before reading further.

Boxes All the Way Down

Let's begin by defining some terms that will make talking about diffy boxes much easier. We call the result of doing one or more iterations on a diffy box a **descendant** of that diffy box. To be concise, we call the result of one iteration on a diffy box its **child** and the original box a **parent** of the new box. Each diffy box has exactly one child, but as we'll see, any box has many possible parents. Furthermore, we say that a box A **generates** a box B if B is a descendant of A. The **zero box** is the box with a zero on each corner, and **order** of a diffy box is the smallest integer n such that the result of n interations on that box is the zero box. The box in *Figure 1* has order 4, the zero box has order 0, any box that does not generate the zero box is defined to have infinite order.

We can now rephrase our original question: does every diffy box have finite order? Since all corners of a diffy box turn nonnegative after one iteration, we don't need to consider diffy boxes with negative entries. This allows us to make a key observation: the largest value in a diffy box never increases. In any diffy box, the minimum possible value of a corner is 0, so the largest possible difference between two adjacent corners is equal to the largest value in the box. If the largest value of the descendants of a diffy box keeps decreasing, it must eventually reach 0. The only way that a diffy box does not generate the zero box is if the largest value of its descendants eventually remains constant.

Suppose we have a diffy box with maximum value M, and its maximum value has been constant over its 3 previous parents. Its parent must have had an M and a 0 sharing an edge - with M as its maximum value, there is no other way to have a difference of M between adjacent corners. The parent of that parent must have had a sequence of

$MM0$ or $M00$ on consecutive corners. Continuing this back one more parent, we get a diffy box containing only M and 0. This shows that if the descendants of a diffy box have a constant maximum value, then they must in fact have only 2 distinct values, M and 0.

Suppose we replace all instances of M in our diffy box with 1. Will this change its order? If M is non-zero, then the answer is no, this simply replaces M with 1 in each of our diffy box's descendants. Therefore, for any diffy box, one of its descendants has the same order as a box containing only the values 1 and 0. Such a diffy box is called a **binary box**.

There are only a finite number of binary boxes, and they are illustrated in *Figure 2*. We can make a directed graph out of binary boxes by drawing an arrow from box A to box B when B is the child of A. As we can see, all binary boxes generate the zero box. To the delight of students everywhere, their tortuous math class subtraction exercises will never keep them at their desks until the end of the period.

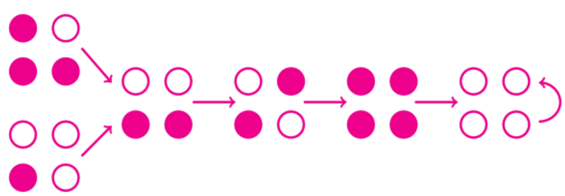

Figure 2: The binary box graph. A filled circle is a 1 and an empty circle is a 0.

To Cycle or not to Cycle?

Squares have failed us, but perhaps if we venture into the world of polygons, we will find one with infinite order. A **diffy n-gon** is an n-sided polygon with numbers on the corners, and to get its child we take the difference of adjacent numbers and put them on the edge joining them.

Looking back on our treatment of diffy boxes, we see that the same arguments show that if our goal is to find an infinite order diffy n-gon, it suffices to only consider **binary n-gons**.

Drawing a graph for diffy triangles the same way we did for squares, we see that there is a non-zero diffy triangle that generates itself. This triangle can never reach the zero triangle, and so we've found what we were looking for! Pentagons also produce a cycle. Just how unlucky did we get by choosing to look at squares first? Exactly which integers n give diffy n-gons with infinite order?

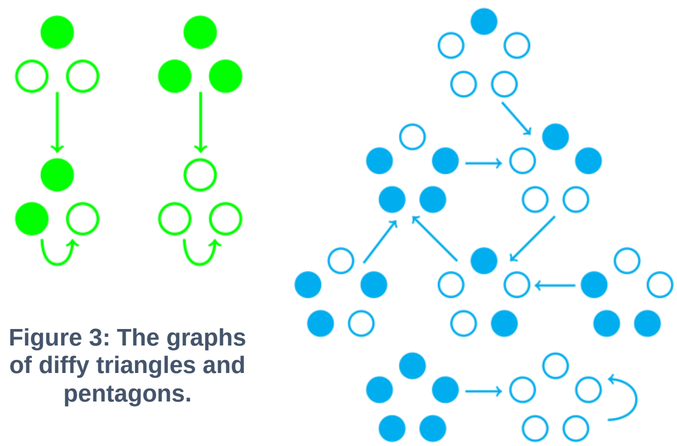

Figure 3: The graphs of diffy triangles and pentagons.

We're looking for **cyclic n-gons**, binary n-gons who generate themselves. We call a binary n-gon **even** if it contains an even number of ones, and **odd** if it contains an odd number of ones. Notice that only even n-gons have parents. This is because a 1 in a child corresponds to a pair 10 or 01 on the corners of its parent. If you start at a corner of an n-gon and move over all the corners in a loop, you end at the same value with which you started, so you must have passed over an even number of these pairs. An extension of this argument shows that any even n-gon must have a parent.

If n is odd, we can always find an even parent for any even n-gon. We have already shown that a parent must exist. If this parent is odd, then we can just swap all the 1's and 0's in the parent. It will have the same child because the 01 pairs are all still in the same places, but now it will be even because the number of corners is odd!

We're now ready to search for cyclic n-gons. Suppose n is odd. If we start with some non-zero even n-gon, we can choose an even parent for it, and then an even grandparent, and so on. There are only a finite number of even n-gons, so at some point we must find a great-great-...-great-grandparent that is one of the polygons we've already seen, and so that polygon is cyclic!

Cyclic n-gons also exist when n is even, as long as n has an odd factor k: one can be constructed by stitching several copies of a cyclic k-gon together. Showing that all diffy n-gons reach the zero n-gon exactly when n is a power of 2 is a fun puzzle, and we encourage you to try it.

Beyond Integers

We've found a solution to our original problem by bending the rules a bit, but our answer isn't entirely satisfying. We wanted a square box with infinite order, and we still haven't found one! Instead of changing the number of sides of our shapes, let's try changing the type of numbers that we choose. Maybe we'll succeed if we allow rational or real numbers in our boxes.

$$\begin{array}{ccc} 6 & - & 5 \\ | & & | \\ 1 & - & 3 \end{array}$$

$$\begin{array}{ccc} 1 & - & 6 \\ | & & | \\ 3 & - & 5 \end{array} \qquad \begin{array}{ccc} 5 & - & 6 \\ | & & | \\ 3 & - & 1 \end{array}$$

$$\begin{array}{ccc} 16 & - & 15 \\ | & & | \\ 11 & - & 13 \end{array} \qquad \begin{array}{ccc} 12 & - & 10 \\ | & & | \\ 2 & - & 6 \end{array}$$

Figure 4: Examples of the four operations.

There are four useful operations we can do on diffy boxes that don't affect their order. We can use (1) rotation and (2) reflection, to rearrange the corners of the box. We can also (3) add the same value to each corner; this doesn't change the difference between adjacent corners, so the child of this modified box is unchanged. Lastly, we can (4) multiply all corners by a nonzero number; we saw this operation earlier when we replaced M with 1 to create a binary box.

Using our operations, we can quickly see that allowing rational numbers isn't useful. Multiplying all corners by the lowest common multiple of their denominators turns our box into an integer diffy box with the same order.

If we allow real numbers, it may seem at first glance that a box with "weird irrational numbers" is almost guaranteed to have infinite order. For example, surely the box with corners π, e, φ, and $\sqrt{2}$ never generates the zero box. In fact, this box has order 5.

We can discover many real-valued diffy boxes that reach the zero box by considering the ordering of the corners. Suppose we list the values of the corners from greatest to least and draw a line between two corners if they are next to each other on this list. Ignoring rotations and reflections, there are only 3 possible pictures we can draw. Any square with a Z or X shape has an order less than 7 (try showing this yourself) so we only need to consider U shapes to find infinite order boxes.

We can cut down on the space of possible squares by converting every square into a standard form. Suppose a diffy box has minimum value m and maximum value M. If we subtract m from each of the corners and then divide all corners by $M - m$, the new

$$\begin{array}{ccc} 4 & - & 3 \\ & / & \\ 2 & - & 1 \end{array}$$

$$\begin{array}{ccc} 4 & & 1 \\ & \times & \\ 2 & - & 3 \end{array} \qquad \begin{array}{ccc} 4 & & 1 \\ | & & | \\ 3 & - & 2 \end{array}$$

Figure 5: The 3 possible shapes: Z, X, and U.

square will have a minimum of 0 and a maximum of 1, but still have the same order. After this transformation is applied, any U-shaped box must have corners 1, y, x, and 0 where $1 \geq y \geq x \geq 0$. Additionally, we can ensure that $y \leq 1 - x$. This is because by multiplying *Figure 6a* by -1 and then adding 1 to all corners, we get *Figure 6b*, and if $y > 1 - x$ in *6a*, then $1 - x < 1 - (1 - y)$ in *6b*.

$$\begin{array}{ccc} 1 & — & 0 \\ | & & | \\ y & — & x \end{array} \qquad \begin{array}{ccc} 1 & — & 0 \\ | & & | \\ 1-x & — & 1-y \end{array}$$

(a) (b)

Figure 6: At least one of these two boxes is in standard form.

Never Quite Reaching Zero

From the graph of standard form boxes in *Figure 7*, we can see that the vast majority have finite order—in fact, if we choose a diffy box at random, there is a 100% chance it has finite order. There is, however, a small region where the boxes have a very high order, and it contains a single standard form box with infinite order. What on earth does this special infinite order box look like?

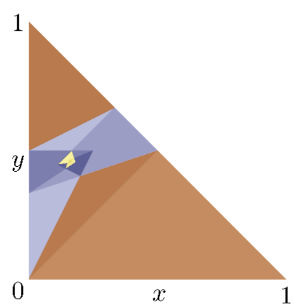

Figure 7: Graph of standard form diffy boxes. The colour of each point (x, y) represents the order of its corresponding box. Brown, purple, and yellow indicate a low, medium, and high order respectively.

The idea is to construct a box such that its child is the same as the original box with all its corners multiplied by a constant r. This way, every descendant will have all its corners multiplied by a power of r, so no descendant is ever the zero box. Let's start by fixing 1 as one of our corners. The corner adjacent to 1 must have value $1 - r$ for their difference to be r. We then want the corner adjacent to that one to be $(1 + r)(1 - r)$ so that their difference is $r(1 - r)$. We can continue this process until we wrap back around to 1 and find that $1 = (1 + r)^3(1 - r)$. The resulting polynomial has a real root, so we've made an infinite order box! This root is about 0.839. If we convert our infinite order box to standard form, we find that y is about 0.4563 and x is about 0.1607.

Venturing Further

We have explored integer diffy n-gons and real-valued diffy boxes, but there are many other directions that the idea of recursively taking differences between numbers can be taken.

Looking at *Figure 3*, we can see that the length and number of different cycles is different depending on n. One question that we can ask about cycles is: Given an integer n, what is the length of the longest cycle of a binary n-gon? The answer to this question is known when certain conditions are put on n, but it is unsolved in general.

We can also move up a dimension to polyhedra. On each iteration, place on the middle of each face the sum of the differences of the adjacent pairs of vertices on that face and connect two of these new numbers if their corresponding faces are adjacent. There is much fun to be had by messing around with diffy boxes, so enjoy yourself!

Thanks to Yuval Wigderson for guiding the authors in their approach to this problem.