1. a) Base Case

$$\frac{1(1+1)(2(1)+1)}{6} = \frac{2\cdot 3}{6} = 1 = 1^2$$

Inductive Step

$$1^2+2^2+\dots+n^2+(n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= (n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right)$$

$$= (n+1)\left(\frac{n(2n+1)+6n+6}{6}\right)$$

$$= (n+1)\left(\frac{n(2n+1)+2(2n+1)+2(n+2)}{6}\right)$$

$$= (n+1)\left(\frac{(n+2)(2n+1)+2(n+2)}{6}\right) = (n+1)\left(\frac{(n+2)(2n+3)}{6}\right)$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \qquad \square$$

b) Base Case

$$\left(1+\frac{1}{1}\right) = 2 = 1+1$$

Inductive Step

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)$$

$$= (n+1)\left(1+\frac{1}{n+1}\right) = (n+1) + \frac{n+1}{n+1} = (n+1)+1$$

2. Let's do induction on n.

Base Case

$(1+x)^1 = 1+x$

Inductive Step

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x)$$
$$= 1+x+nx+nx^2 = 1+(n+1)x+nx^2 > 1+(n+1)x. \quad \square$$

3. Base Case

2 is a prime

Inductive step

Consider a number n. If n is prime, then it already satisfies the condition. If n is composite, then it is a multiple of integers between 2 and n. Since each of those numbers are a product of primes, n is a product of primes. $\square$

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Induction

4. a) Regular induction won't work because we are not assuming that $F_{n-1} < (\frac{7}{4})^{n-1}$. The best we can say is $F_{n-1} < (\frac{7}{4})^n$, which only allows us to prove that:

$$F_{n+1} = F_n + F_{n-1} < \left(\frac{7}{4}\right)^n + \left(\frac{7}{4}\right)^n = 2\left(\frac{7}{4}\right)^n = \frac{8}{7}\left(\frac{7}{4}\right)^{n+1}$$

This a slightly worse bound than $F_{n+1} < (\frac{7}{4})^{n+1}$.

b) The key step of the proof is that $1 + \frac{7}{4} \leq (\frac{7}{4})^2$. This is the only step that limits the size of β, so β is the largest number where $1 + \beta \leq \beta^2$. β is smallest when the two expressions are equal, so we just have to solve the equation $1 + \beta = \beta^2$. Using the quadratic formula, we get:

$$\beta = \frac{1 \pm \sqrt{5}}{2} = 1.618\ldots \text{ or } -0.618\ldots$$

We want a positive value for β, so $\beta = \frac{1+\sqrt{5}}{2}$. Now let's check that this value works for the first two Fibonacci numbers.

$F_1 = 1 < 1.618\ldots = \beta$
$F_2 = 1 < 2.618\ldots = \beta^2$

Therefore, $\beta = \frac{1+\sqrt{5}}{2}$. $\square$

(This is known as the golden ratio, or $\varnothing$)

c) First, let's find the largest value of c such that $c\beta^n \leq F_n$ for the first two Fibonacci numbers. c will be largest when they are equal.

F_1	F_2
$c\beta^1 = F_1 = 1$	$c\beta^2 = F_2 = 1$
$c\varnothing = 1$	$c\varnothing^2 = 1$
$c = 1/\varnothing$	$c = 1/\varnothing^2$

So, our largest possible value of c is $1/\phi^2$. Now, lets use induction to show that this will work for the nth Fibonacci number.

Inductive Step

$$
\begin{aligned}
&F_{n+1} \\
&= F_n + F_{n-1} \\
&\geq c\beta^n + c\beta^{n-1} \\
&= c\beta^{n-1}(\beta + 1) \\
&= c\beta^{n-1} \cdot \beta^2 \\
&= c\beta^{n+1}
\end{aligned}
$$

which is what we wanted. Therefore, the largest possible value of c is $1/\phi^2 \approx 0.382$. □

Table of values

n	β^n	F_n	$c\beta^n$
1	1.618	1	0.618
2	2.618	1	1
3	4.24	2	1.618
4	6.85	3	2.618
5	11.09	5	4.24
6	17.94	8	6.85
7	29.03	13	11.09

Always verify your solutions for small values!