Problem Solving Club
Combinatorial Game Theory Solutions

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Problem 1

The rules of NIM are changed so that 1, 3, or 4 stones can be removed each turn instead of 1, 2, or 3. Call this new game (1,3,4)NIM. For the initial values of 30, 31, 32, 33, 34, and 35 stones in the pile, when does Player 2 have a winning strategy?

Solution There is a pattern of N's and P's that repeats every 7 values. Namely, a pile is P if and only if it is a multiple of 7 or 2 more than than a multiple of 7. We can prove this by induction.

Base Case:

0 is P. 1 is N since it can move to 0. 2 is P since it can only go to 1, which is N. 3,5, and 6 are N because they can move to 2, and 4 is N because it can move to 0.

Inductive Step:

Assume the pattern holds for all integers less than n, where n is a multiple of 7. Our gal is to show that the pattern holds for the 7 integers from n to $n+6$. n can only go to $n-1$, $n-3$, and $n-4$, which are all N by the inductive hypothesis (the closest P pile to n is $n-5$). So, n is P. This means that $n+1$, $n+3$, and $n+4$ are all N. $n+2$ can only move to $n+1$, $n-1$, and $n-2$, which are all N, so $n+2$ is P. This means that $n+5$ and $n+6$ are N. This completes the proof.

30 is 2 more than a multiple of 7, and 35 is a multiple of 7. None of the others fit this criterion. So, of the given numbers of stones, 30 and 35 are the only ones for which Player 2 has a winning strategy.

Problem 2

NIM is played with 2 piles of 10 stones each. On their turn, each player chooses one pile to remove 1,2, or 3 stones. Which player wins now?

Solution Player 2 has a winning strategy here. Whatever number of stones Player 1 takes from one of the piles, Player 2 can just take the same number of stones from the other pile. This ensures that after Player 2's turn the number of stones in each pile is equal, and after Player 1's turn the number of stones in each pile is never equal. At the end of the game, there are an equal number of stones in each pile, that number being 0. So, this must have happened after Player 2's turn, meaning Player 2 won.

Problem 3

On each turn, a player can either remove 1 stone from the pile or divide the number of stones in the pile by 2 (rounding down). Starting with 100 stones, which player wins?

Solution We will figure out if 100 is N or P by starting with small numbers of stones and working up. 0 is P, since the previous player just won. 1 is N since it's possible to move to 0. 2 is P since it can only move to 1. 3, 4, and 5 are all N since they can move to 2. 6 is N since it can only move to 3 or 5, both of which are N. 7 is N since it can move to 6. 8 is P since it can only move to 4 or 7, both of which are N. 9 is N since it can move to 8. 10 is P since it can only move to 9 or 5, both of which are N. 11 is N since it can move to 10, and 12 is N since it can move to 6.

Now, 21 is N because it can move to 10. 22 is P since it can only move to 21 or 11, both of which are N. 23 is N because it can move to 22. 24 is P since it can only move to 23 or 12, both of which are N. 25 is N because it can move to 24.

49 is N because it can move to 24, which is P. 50 is P because it can only move to 49 or 25, both of which are N. Finally, 100 is N because it can move to 50. So, Player 1 wins.