Problem Solving Club
Infinite Series Solutions

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1 Problem 1

What is the value of

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \ldots? \tag{1.1}$$

Solution This can be solved using the formula proved in the presentation,

$$1 + r + r^2 + r^3 + ... = \frac{1}{1-r} \tag{1.2}$$

since (1.1) is equal to

$$-1 + \left(1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \ldots\right). \tag{1.3}$$

Using (1.2) on the bracketed part of (1.3) (assuming $|\frac{1}{n}| < 1$) gives

$$\begin{aligned} & -1 + \left(\frac{1}{1-\frac{1}{n}}\right) \\ &= -1 + \frac{1}{\frac{n-1}{n}} \\ &= -1 + \frac{n}{n-1} \\ &= \frac{1}{n-1}. \end{aligned}$$

If $|\frac{1}{n}| \geq 1$, or equivalently, if $|n| < 1$, then (1.1) goes to ∞ since each of its terms is ≥ 1.

2 Problem 2

Use a similar method to the one shown to find a closed-form expression for

$$1 + r + r^2 + \ldots + r^n. \tag{2.1}$$

Use the value of this expression as $n \to \infty$ to find the value of $1 + r + r^2 + \ldots$ How does this method explain the restriction that $|r| < 1$?

Solution Let's proceed in the same way as in the presentation. First let S be the value of (2.1). It follows that

$$rS = r + r^2 + r^3 + \cdots + r^{n+1}. \tag{2.2}$$

Subtracting (2.2) from (2.1) gives

$$\begin{aligned} S - rS &= 1 + (r - r) + (r^2 - r^2) + \ldots + (r^n - r^n) - r^{n+1} \\ S(1 - r) &= 1 - r^{n+1} \\ S &= \frac{1 - r^{n+1}}{1 - r}. \end{aligned} \tag{2.3}$$

Now, if $|r| < 1$, then as $n \to \infty$, $r^{n+1} \to 0$. This means that in the limit, we can remove the r^{n+1} term from (2.3), leaving us with (1.2). On the other hand, if $|r| > 1$, then $|r^{n+1}| \to \infty$ as $n \to \infty$, so it cannot be removed from the equation in the limit. Finally, we can check that if $|r| = 1$ then the sum will not converge as we add more terms, so (1.2) holds if and only if $|r| < 1$.

3 Problem 3

About how many factors of 5 would you expect $n!$ to have when n is large?
Ex: 25! has 6 factors of 5. Give your answer in terms of n.

Solution $n!$ is all the positive integers $\leq n$ multiplied together, so we have to find about how many factors of 5 there are among all the positive integers $\leq n$. About 1/5 of positive integers are divisible by 5, so there should be about $n/5$ factors of 5 in the integers $\leq n$. But we aren't counting integers divisible by 5^2 here, for example 25 contributes 2 factors of 5 to $n!$, but only gets counted once. So we have to add another term of $n/5^2$ to keep our approximation accurate. but now we aren't counting the numbers that are divisible by 5^3, so we have to add another term of $n/5^3$. In this way, we will keep adding terms to the approximation until we reach the final approximation of

$$\frac{n}{5} + \frac{n}{5^2} + \frac{n}{5^3} + \dots$$

Using our answer from Problem 1, we can see that this is equal to $n/4$, which is our final approximation. So, we can say that 1000000! has about 250000 factors of 5. (The actual number is 249998)

Bonus Problem Try solving Problem 3 for the number of factors of prime numbers other than 5, like 2, 3, or 7. Can you come up with a general solution for all prime numbers? Now try solving Problem 3 for composite numbers. Is this different than solving it for prime numbers? (Hint: It is, but how?)