Problem Solving Club
Combinatorics Solutions

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Problem 1 Esteban has a strawberry, a blueberry, a raspberry and a blackberry, and he wants to place them on the corners of a cake. How many ways are there to place the berries if:

a) He wants to place 1 berry at each corner.

b) He wants to place 1 or 0 berries at each corner.

c) He wants to place 1 berry at each corner, but he has 4 berries of each type instead of 1.

a) This question is just asking the number of ways to arrange 4 objects, which is $4! = 24$.

b) Esteban can have either 4, 3, 2, 1, or 0 berries on the cake. Let's find the number of possibilities in each case and add them up.

Case 1: There are 4 berries

This is the situation from part 1, so there are $4! = 24$ possibilities.

Case 2: There are 3 berries

There are $\binom{4}{3}$ ways to select the berries Esteban uses. There are $\binom{4}{3}$ ways to choose which three corners to place those berries on. Finally, there are 3! ways to arrange those berries on the three chosen corners. So, the total number of possibilities is

$$\binom{4}{3}\binom{4}{3}3! = (4)(4)(6) = 96.$$

Case 3: There are 2 berries

There are $\binom{4}{2}$ ways to select the berries Esteban uses. There are $\binom{4}{2}$ ways to choose which two corners to place those berries on. Finally, there are 2! ways to arrange those berries on the two chosen corners. So, the total number of possibilities is

$$\binom{4}{2}\binom{4}{2}2! = (6)(6)(2) = 72.$$

Case 4: There is 1 berry

There are 4 choices for the berry Esteban uses, and there are 4 corners the berry could be placed on. So, the total number of possibilities is

$$(4)(4) = 16.$$

Case 5: There are no berries

There is one way to place 0 berries on the cake.

So, the total number of ways Esteban can place the berries is

$$24 + 96 + 72 + 16 + 1 = 209.$$

c) There are 4 choices of berry for each of the 4 corners, so the total number of possibilities is

$$4^4 = 256.$$

Problem 2 Jason has a full suit of 13 hearts, if he picks 3 cards, what's the probability he gets:

a) The Ace

b) The King and Queen

c) 4, 5, and 6

a) There are $\binom{13}{3} = 286$ ways to choose 3 cards from the deck. Once the Ace is chosen, there are $\binom{12}{2} = 66$ ways to choose 2 other cards from the deck. So, the probability of choosing a hand containing an the Ace is $66/286 = 3/13 \approx 23.08\%$.

b) Once the King and Queen are chosen, there are 11 ways to choose another card from the deck. So, the probability of choosing a hand containing the King and Queen is $11/286 = 1/26 \approx 3.85\%$.

c) There is only one way to choose the 4, 5, and 6 from the deck. So, the probability of choosing that hand is $1/286 \approx 0.35\%$.

Problem 3 Prove that

$$\binom{2022}{0} + \binom{2022}{2} + \binom{2022}{4} + ... + \binom{2022}{2020} + \binom{2022}{2022} = 2^{2021}$$

Lemma 1.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof.

$$
\begin{aligned}
&\binom{n-1}{k-1} + \binom{n-1}{k} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{(n-1)!(k + (n-k))}{k!(n-k)!} \\
&= \frac{(n-1)!(n)}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}
$$

□

Problem 3 Solution

Since

$$\binom{2022}{0} = 1 = \binom{2021}{0} \quad \text{and} \quad \binom{2022}{2022} = 1 = \binom{2021}{2021},$$

we can rewrite the sum as

$$
\begin{aligned}
&\binom{2022}{0} + \binom{2022}{2} + \binom{2022}{4} + ... + \binom{2022}{2020} + \binom{2022}{2022} \\
&= \binom{2021}{0} + \binom{2022}{2} + \binom{2022}{4} + ... + \binom{2022}{2020} + \binom{2021}{2021}.
\end{aligned}
$$

Now, using Lemma 1 on the remaining terms containing a 2022, we get

$$
\begin{aligned}
&\binom{2021}{0} + \binom{2022}{2} + \binom{2022}{4} + ... + \binom{2022}{2020} + \binom{2021}{2021} \\
&= \binom{2021}{0} + \binom{2021}{1} + \binom{2021}{2} + ... + \binom{2021}{2019} + \binom{2021}{2020} + \binom{2021}{2021}.
\end{aligned}
$$

This is just the total number of subsets of 2021 objects. For each subset, each of the 2021 objects is either in the subset or it isn't. Since there are 2 possibilities for each object, there are 2^{2021} subsets in total, which is exactly what was required.