1 December 2021

Problem Solving Club

Proof by Contradiction

Number Systems

Saying “x is an integer and y is a real number” is the same as saying “$x \in \mathbb{Z}$ and $y \in \mathbb{R}$”

Natural	Zahlen (German for numbers)	Quotient	Real	Complex
$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
Naturals: 1 23 4005	Integers: 0 23 -200	Rational: ½, 355/133	Real: $-\pi$ 25	Complex: 2+3i

A Prime Problem

Are there infinitely many prime numbers?

This problem is hard to solve directly because there are infinitely many integers to check.

Induction won't help us because there is no clear connection between prime n and prime n+1.

Proof by Contradiction

Proof by Contradiction is a proof method where, in order to prove that a statement is true, you assume that it is false and then find that that assumption leads to a contradiction.

A contradiction is some false statement like “0=1” or “2 is odd”.

By proving that a statement is not false, you are proving that it is true!

Prime Problem

Let's prove that there are infinitely many primes using a proof by contradiction.

Assume that there are finitely many primes.

Call the total number of primes n. Since there are finitely many primes, we can write them all out in a list like this:

$p_1, p_2, p_3, \ldots, p_{n-1}, p_n$

Showing that there is a prime number that is not on this list would be a contradiction, since we defined the list have all the primes. Now, consider the number:

$N = p_1 p_2 p_3 \ldots p_{n-1} p_n + 1$

There are 2 options, either N is prime or N is composite.

Prime Problem

$N = p_1 p_2 p_3 \cdots p_{n-1} p_n + 1$

Case 1: N is prime

N has to be bigger than all the primes on the list, so N is not on the list. That's a contradiction, so N can't be prime.

Case 2: N is composite

Since N is composite, it must be a product of primes, as we showed last week.

However, N is not divisible by any of the primes on the list. So, N must be divisible by some prime not on the list. That's a contradiction, so N can't be composite.

There's a contradiction in either case, so our original assumption that there is finitely many primes must be wrong. There must be infinitely many primes.

Week 9 - Dec 1

1. Prove that if n^2 is odd, then n is odd.
2. Prove that the sum of a rational number and an irrational number is irrational.
3. Prove that there is no smallest positive rational number.
4. Prove that there are no integers a and b where $18a + 6b = 1$.
5. Prove that if an integer n is composite, it has a prime factor $\leq \sqrt{n}$.

(This week was a bit dry, but hopefully next week will be more exciting)