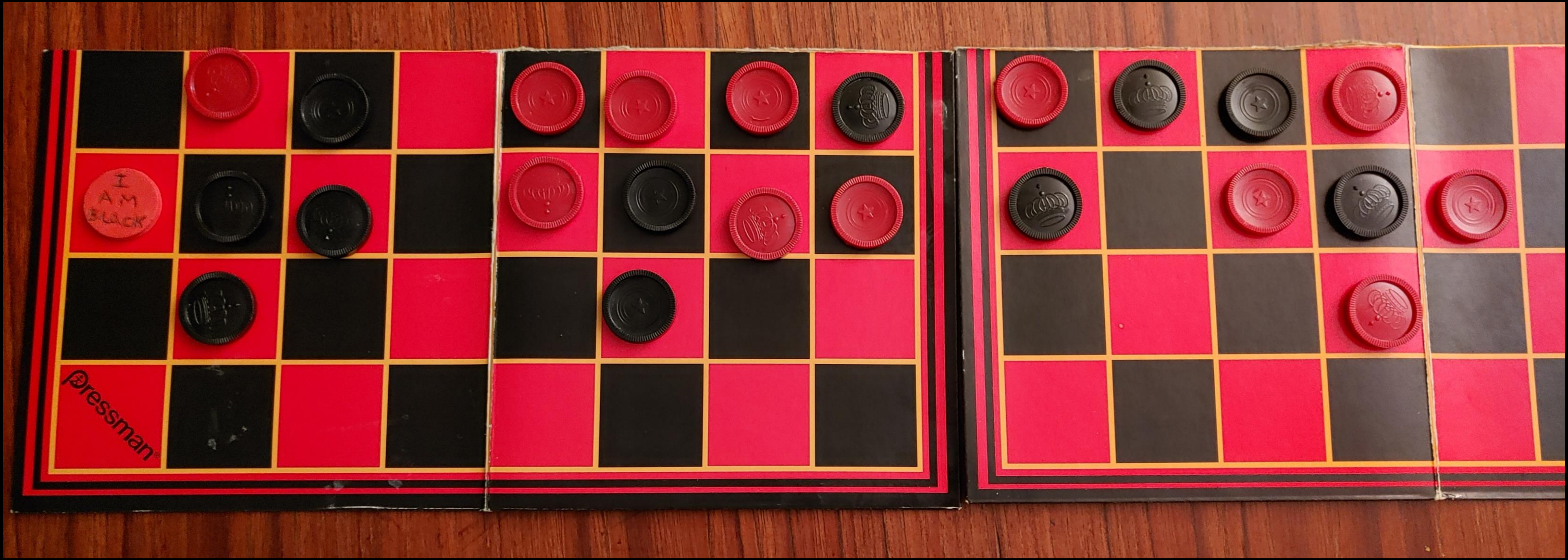
25 November 2021

Problem Solving Club

Induction

Conway Checkers with Diagonal Jumps

My best attempt: Row 6 in 24 checkers



Conway Checkers with Diagonal Jumps

The theoretical best:

TABLE 1. Unreachable levels of DIAPEGS in $\mathbb{Z}^n$

n	unreachable level k^*	$f(n,k^*)$	$f(n,k^*-1)$
1	2	1.000	1.618
2	9	0.877	1.308
3	18	0.872	1.286
4	28	0.967	1.424
5	40	0.689	1.017
6	51	0.932	1.376
7	64	0.703	1.041

However, using ad hoc methods we have constructed a game in $\mathbb{Z}^2$ that reaches level 8, so here the upper bound is sharp! It is possible that the bounds are sharp for all n, but completely new construction ideas would be needed to prove such a result.

A Difficult Fibonacci Problem

Let F_n represent the nth Fibonacci number. So $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. Prove that for every positive integer n, $F_n < (7/4)^n$.

- This problem is hard because we have to prove this for infinitely many values of n
- Directly finding a pattern won’t help us here

Induction

Mathematical Induction is a powerful tool for proving a statement for infinitely many values of some variable.

Imagine that $P(n)$ is some statement that we want to prove for all positive integers n. $P(n)$ could be "$F_n < (7/4)^n$" or "$1 + 2 + \ldots + n = n(n+1)/2$" or anything else.

Induction consists of two steps:

Base Case: Show that $P(1)$ is true

Inductive Step: Show that if $P(m)$ is true, then $P(m+1)$ is true.

Can you see why this proves $P(n)$ for all positive integers?

Induction

A common analogy for induction is a string of dominoes. We can prove that all the dominoes will fall over by proving that the first domino falls over (Base Case) and that if a domino falls over, it will knock over the next domino (Inductive Step).

Example:

Let's prove that $1 + 2 + \ldots + n = n(n+1)/2$ using induction.

Base Case

$1(1+1)/2 = 2/2 = 1$

Inductive Step

Start with:

$1 + 2 + \ldots + m + (m+1)$

Since we're assuming that $P(m)$ is true, we can write:

$= m(m+1)/2 + (m+1)$

Now just use algebra to finish the proof.

Example:

$$m(m+1)/2 + (m+1)$$

$$= (m+1)(m/2 + 1)$$

$$= (m+1)((m+2)/2)$$

$$= (m+1)((m+1)+1)/2$$

which is exactly what we wanted!

Therefore,

$$1 + 2 + \ldots + n = n(n+1)/2$$

for all positive integers n.

A Quick Note

It is said that induction is only used to confirm guesses. Induction allowed us to prove the previous theorem, but it gave us no idea where $n(n+1)/2$ came from.

You could have made that guess by finding a pattern in the sums from 1 to n.

Strong Induction

Regular Induction still isn't enough to solve the Fibonacci problem, we need strong induction. Strong Induction is almost the same as regular induction with one slight change…

Induction consists of two steps:

Base Case: Show that P(1) is true

Inductive Step: Show that if P(m) is true, then P(m+1) is true.

Strong Induction consists of two steps:

Base Case: Show that P(1) is true

Inductive Step: Show that if P(1), P(2), … P(m-1), and P(m) are true, then P(m+1) is true.

Fibonacci Problem

Let F_n represent the nth Fibonacci number. So $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. Prove that for every positive integer n, $F_n < (7/4)^n$.

Base Cases

$(7/4)^1 = 7/4 > F_1$

$(7/4)^2 = 49/16 > F_2$

Fibonacci Problem

Inductive Step

F_{n+1}

$= F_n + F_{n-1}$

$< (7/4)^{n} + (7/4)^{n-1}$

$= (7/4)^{n-1} \cdot (7/4 + 1)$

$< (7/4)^{n-1} \cdot (7/4)^{2}$

$= (7/4)^{n+1}$

Which is what we wanted!

The second to last step is true because

$7/4 + 1 = 11/4 < 3 < 49/16 = (7/4)^{2}$

Week 8 - Nov 24

1. Prove the following relations using induction:

a) $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

b) $(1+1/1)(1+1/2)\ldots(1+1/n) = n+1$

2. Show that if n is a positive integer and x is a real number greater than -1, then $(1+x)^n \geq 1 + nx$.

3. Show that every integer >1 can be represented as the product of one of more primes.

4. Reexamine the Fibonacci proof.

a) Why does regular induction not work in this case?

b) What is the smallest positive value of β where you can prove that $F_n < \beta^n$? (It's lower than 7/4)

c) Once you've found β, what is the largest value of c you can find such that $F_n \geq c\beta^n$?