21 October 2021

Problem Solving Club

Divisibility

Divisibility

Divisibility is one of the foundations of Number Theory.

> n is divisible by d if $n = qd$ for some integer q.

10 is divisible by 5 since $10 = 5 \times 2$

10 is divisible by -2 since $10 = -2 \times -5$

10 is **not** divisible by 3 because there is no integer q where $10 = 3q$

Representing a number by its digits

100s 10s 1s

256

$256 = 6 + 10(5) + 100(2)$

100s 10s 1s

$\dots d_2 d_1 d_0$

$N = d_0 + 10d_1 + 100d_2 + 1000d_3\dots$

Divisibility by 1

Every integer is divisible by 1

Divisibility by 2

The last digit is even.

0 2 4 6 8

This is because 10, 100, 1000,… are all divisible by 2.

$$N = d_0 + 10d_1 + 100d_2 + 1000d_3 \ldots$$

The only digit that determines whether N is divisible by 2 is d_0.

Divisibility by 4

The last 2 digits are divisible by 4.

237,239,236

This is because 100, 1000, 10000… are all divisible by 4.

$$N = d_0 + 10d_1 + 100d_2 + 1000d_3\ldots$$

The only digits that determine whether N is divisible by 4 are d_0 and d_1.

Divisibility by 8

The last 3 digits are divisible by 8.

237,239,236

This is because 1000, 10000, 100000…
are all divisible by 8.

$$N = d_0 + 10d_1 + 100d_2 + 1000d_3\ldots$$

The only digits that
determine whether N is
divisible by 8 are d_0, d_1,
and d_2.

Divisibility by 5

The last digit is 5 or 0.

237,239,235

This is because 10, 100, 1000,… are all divisible by 5.

$N = d_0 + 10d_1 + 100d_2 + 1000d_3 \ldots$

The only digit that determines whether N is divisible by 5 is d_0.

Divisibility by 10

The last digit is 5 or 0.

237,239,230

This is because 10, 100, 1000,… are all divisible by 10.

$N = d_0 + 10d_1 + 100d_2 + 1000d_3\dots$

The only digit that determines whether N is divisible by 10 is d_0.

1 2 3 4 5 6 7 8 9 10 11 12

Divisibility by 6

The number passes the divisibility tests for 2 and 3.

$$6 = 2 \times 3$$

1 2 3 4 5 6 7 8 9 10 11 12

Divisibility by 12

The number passes the divisibility tests for 4 and 3.

$$12 = 4 \times 3$$

1 2 3 4 5 6 7 8 9 10 11 12

Divisibility by 3, 7, 9 and 11.

These ones are harder to explain than the others

1 2 3 4 5 6 7 8 9 10 11 12

Divisibility by 3 & 9

3: The sum of digits of N is divisible by 3.
9: The Sum of digits of N is divisible by 9.

723,129 Sum of Digits = 7 + 2 + 3 + 1 + 2 + 9

= 24

This number is divisible by 3, but not by 9

Proof of divisibility by 3 & 9

Let d(N) be the sum of digits of N.

If $N = d_0 + 10d_1 + 100d_2 + 1000d_3 \ldots$
then $d(N) = d_0 + d_1 + d_2 + d_3 \ldots$

Consider the value of N − d(N):

$N - d(N)$
$= (d_0 + 10d_1 + 100d_2 + 1000d_3 \ldots) - (d_0 + d_1 + d_2 + d_3 \ldots)$
$= d_0 - d_0 + 10d_1 - d_1 + 100d_2 - d_2 + \ldots$
$= 9d_1 + 99d_2 + 999d_3 \ldots$
$= 9(d_1 + 11d_2 + 111d_3 \ldots)$

So N − d(N) is a multiple of 9. If d(N) is a multiple of 9, then N must be a multiple for N − d(N) to still be a multiple of 9. This is what we wanted to show. Since 9 is a multiple of 3, this also proves the rule for 3.

Divisibility by 11

The alternating sum of digits of N is divisible by 3.

723,129

Alternating Sum of Digits = 7 - 2 + 3 - 1 + 2 - 9

= 0

0 is a multiple of 11, so 723,129 is divisible by 11.

Proof of divisibility by 11

Let s(N) be the alternating sum of digits of N.

If $N = d_0 + 10d_1 + 100d_2 + 1000d_3 \ldots$
then $s(N) = d_0 - d_1 + d_2 - d_3 \ldots$

Consider the value of N − d(N):

$N - s(N)$
$= (d_0 + 10d_1 + 100d_2 + 1000d_3 \ldots) - (d_0 - d_1 + d_2 - d_3 \ldots)$
$= d_0 - d_0 + 10d_1 + d_1 + 100d_2 - d_2 + \ldots$
$= 11d_1 + 99d_2 + 1001d_3 \ldots$
$= 11(d_1 + 9d_2 + 91d_3 \ldots)$

So N − s(N) is a multiple of 11. If s(N) is a multiple of 11, then N must be a multiple of 7 for N − s(N) to still be a multiple of 11. This is what we wanted to show.

Divisibility by 7

If we write N as Ad = 10A + d, then N is divisible by 7 if and only if A − 2d is divisible by 7.

723,128 = 10(72,312) + 8

723,128 is divisible by 7 if and only if 72,312 − 2(8) = 72296 is divisible by 7.

72296 is divisible by 7 if and only if 7229 − 2(6) = 7217 is divisible by 7.

7217 is divisible by 7 if and only if 721 − 2(7) = 707 is divisible by 7.

707 is divisible by 7, so 7,217, 72,296, 723,128 are all divisible by 7.

Proof of divisibility by 7

Let N = $\underline{Ad}$ = $10A + d$.
Let $s = A - 2d$…

Consider the value of $N - 3s$:

$$
\begin{aligned}
&N - 3s \\
&= (10A + d) - 3(A - 2d) \\
&= 7A + 7d \\
&= 7(A+d)
\end{aligned}
$$

So N − s is a multiple of 7. If s is a multiple of 7, then N must be a multiple of 7 for N − s to still be a multiple of 7. This is what we wanted to show.

Divisibility Rules 1-12

1: Every Integer
2: Last digit is even
3: Sum of Digits is divisible by 3
4: Last 2 digits are divisible by 4
5: Last digit is 5 or 0
6: Divisible by 2 and 3
7: Rest of the number minus twice the last digit is divisible by 7
8: Last 3 digits are divisible by 8
9: Sum of Digits is divisible by 9
10: Last digit is 0
11: Alternating sum is divisible by 11
12: Divisible by 4 and 3

Week 4 - Oct 20

1. There are 24 4-digit numbers using the digits 1-4 once each. How many are divisible by 4?

2. 5,41G,507,2H6 is divisible by 72. What are all possible pairs of digits (G,H)?

3. A3,640,57B is divisible by 99. What are all possible pairs of digits (A,B)?

4. Create a divisibility test for 13 based on the one shown for 7.

5. Try to complete as many problems as you can from a previous COMC contest. (Link posted on GC)

No meeting next week since the COMC would be happening on the same day.