9 February 2022

Problem Solving Club

Combinatorial Game Theory

NIM

NIM starts with a pile of stones.

Players take turns removing stones from the pile, with Player 1 going first and Player 2 going second.

On each turn, a player can remove 1, 2, or 3 stones. Whoever takes the last stone wins.

If we assume that both players play optimally, then who wins if there are 12 stones in the initial pile?

Combinatorial Games

A combinatorial game is a game where two players take turns moving making moves, and each player has full information of the state of the game.

Combinatorial Games:

- Tic Tac Toe
- Chess
- NIM

Non-Combinatorial Games:

- Uno
- Monopoly
- Poker

Combinatorial Game Theory studies the properties of combinatorial games.

NIM is a particularly simple Combinatorial Game because it is symmetric: each player has the exact same set of possible moves.

Combinatorial Positions

In a combinatorial game, a position is an N position if the next player to move has a winning strategy. (N for Next)

A position is a P position if the previous player to move has a winning strategy. (P for Previous)

(1) An N position is one where it is possible to move to a P position.

(2) A P position is one where it is only possible to move to N positions.

Analysing NIM

0 1 2 3 4 5 6 7 8 9 10 11 12

We will colour a position green if it is an N position and red if it is a P position.

(1) An N position is one where it is possible to move to a P position.

(2) A P position is one where it is only possible to move to N positions.

Analysing NIM

0 1 2 3 4 5 6 7 8 9 10 11 12

At 0 stones in the pile, the previous player just won, so 0 is P.

At one stone, the next player can move to 0, so 1 is N due to (1).

2 and 3 can also move to 0, so they are also N.

(1) An N position is one where it is possible to move to a P position.

(2) A P position is one where it is only possible to move to N positions.

Analysing NIM

0 1 2 3 4 5 6 7 8 9 10 11 12

4 can only move to 1, 2, or 3, so 4 is a P position by (2).

5, 6, and 7 can move to 4, so they are all N.

8 will be P since it can only move to 5, 6, or 7.

(1) An N position is one where it is possible to move to a P position.

(2) A P position is one where it is only possible to move to N positions.

Analysing NIM

0 1 2 3 4 5 6 7 8 9 10 11 12

At this point, you can see that the pattern will repeat, with every multiple of 4 being P.

12 is P, meaning that the second player will win if the game starts with 12 stones.

(1) An N position is one where it is possible to move to a P position.

(2) A P position is one where it is only possible to move to N positions.

Analysing NIM

0 1 2 3 4 5 6 7 8 9 10 11 12

In order to win, the second player just needs to make the number of stones a multiple of 4 on each of their turns.

(1) An N position is one where it is possible to move to a P position.

(2) A P position is one where it is only possible to move to N positions.

Week 16 – Feb 9

1. The rules of NIM are changed so that 1, 3, or 4 stones can be removed each turn instead of 1, 2, or 3. Call this new game (1,3,4)NIM. For the initial values of 30, 31, 32, 33, 34, and 35 stones in the pile, when does Player 2 have a winning strategy?

2. NIM is played with 2 piles of 10 stones each. On their turn, each player chooses one pile to remove 1,2, or 3 stones. Which player wins now?

3. On each turn, a player can either remove 1 stone from the pile or divide the number of stones in the pile by 2 (rounding down). Starting with 100 stones, which player wins?