26 January 2022

Problem Solving Club

Infinite Series

Powers of 1/2

$1 = 1$

$1 + 1/2 = 1.5$

$1 + 1/2 + 1/4 = 1.75$

$1 + 1/2 + 1/4 + 1/8 = 1.875$

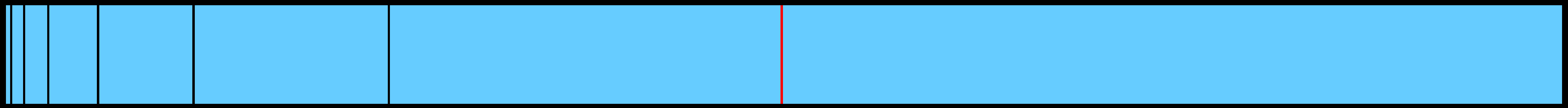
$1 + 1/2 + 1/4 + 1/8 + 1/16 = 1.9375$

$1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 1.96875$

…

$1 + 1/2 + \ldots + 1/2^{10} = 1.9990234\ldots$

Powers of 1/2



Does it make sense to say that

$1 + 1/2 + 1/4 + \ldots = 2$?

Shouldn't infinitely many numbers add to something infinitely big?

How do we know that this series is equal to 2 and not 2.0000001 or 1.999999?

Powers of r

$$S = 1 + r + r^2 + r^3 + \dots$$

$$rS = r + r^2 + r^3 + r^4 + \dots$$

$$S - rS = 1 + (r-r) + (r^2-r^2) + \dots$$

$$S - rS = 1$$

$$S(1-r) = 1$$

$$S = 1/(1-r)$$

Powers of r

$1 + r + r^2 + r^3 + \ldots = 1/(1-r)$

Disclaimer: Only works when $|r| < 1$

Luckily, $|1/2| < 1$

$1 + 1/2 + 1/4 + \ldots$

$= 1/(1-(1/2))$

$= 1/(1/2)$

$= 2$

Like we expected!

Harmonic Series

$1 = 1$

$1 + 1/2 = 1.5$

$1 + 1/2 + 1/3 = 1.8333\ldots$

$1 + 1/2 + 1/3 + 1/4 = 2.0833\ldots$

$1 + 1/2 + 1/3 + 1/4 + 1/5 = 2.2833\ldots$

$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 = 2.45$

…

$1 + 1/2 + \ldots + 1/30 = 3.99498\ldots$

Harmonic Series

It doesn't approach any number,
it just keeps getting bigger.

$1 + 1/2 + \ldots + 1/31 = 4.027\ldots$

But how can we prove this?

Harmonic

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + \ldots$$

$$> 1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \ldots$$

$$= 1 + 1/2 + 1/2 + 1/2 + \ldots$$

$$\Rightarrow \infty$$

Since the Harmonic Series is larger than something that goes to infinity, it must also go to infinity

Week 15 - Jan 26

1. What is the value of $1/n + 1/n^2 + 1/n^3 + \ldots$?

2. Use a similar method to the one shown to find a closed-form expression for $1 + r + r^2 + \ldots + r^n$.
Use the value of this expression as $n \to \infty$ to find the value of $1 + r + r^2 + \ldots$ How does this method explain the restriction that $|r| < 1$?

3. About how many factors of 5 would you expect $n!$ to have when n is large? Ex: $25!$ has 6 factors of 5. Give your answer in terms of n.

Hint: Your answer should somehow relate to an infinite series.