19 January 2022

Problem Solving Club

Combinatorics

Rearranging Stuff

How many ways are there to rearrange 3 objects, like the numbers 1 - 3?

123 works, so does 213.

There are 6:

123
132
213
231
312
321

Here's how we could figure this out systematically:

1, 2, or 3	One of the two digits not chosen yet	The only digit not chosen yet
3	2	1

$3 \times 2 \times 1 = 6$

Rearranging Stuff

The same logic works for any number of objects:
The number of ways to rearrange 5 objects is $5 \times 4 \times 3 \times 2 \times 1$.
The number of ways to rearrange 100 objects is $100 \times 99 \times \ldots \times 3 \times 2 \times 1$.

Multiplying all the numbers from 1 to some other number is called a factorial, and is denoted using an exclamation mark (!).

Factorials

$0! = 1$

$1! = 1$

$2! = 1 \times 2 = 2$

$3! = 1 \times 2 \times 3 = 6$

$4! = 1 \times 2 \times 3 \times 4 = 24$

$5! = 1 \times 2 \times \ldots \times 5 = 120$

$6! = 1 \times 2 \times \ldots \times 6 = 720$

$7! = 1 \times 2 \times \ldots \times 7 = 5{,}040$

$8! = 1 \times 2 \times \ldots \times 8 = 40{,}320$

$9! = 1 \times 2 \times \ldots \times 9 = 362{,}880$

$10! = 1 \times 2 \times \ldots \times 10 = 3{,}628{,}800$

$11! = 1 \times 2 \times \ldots \times 11 = 39{,}916{,}800$

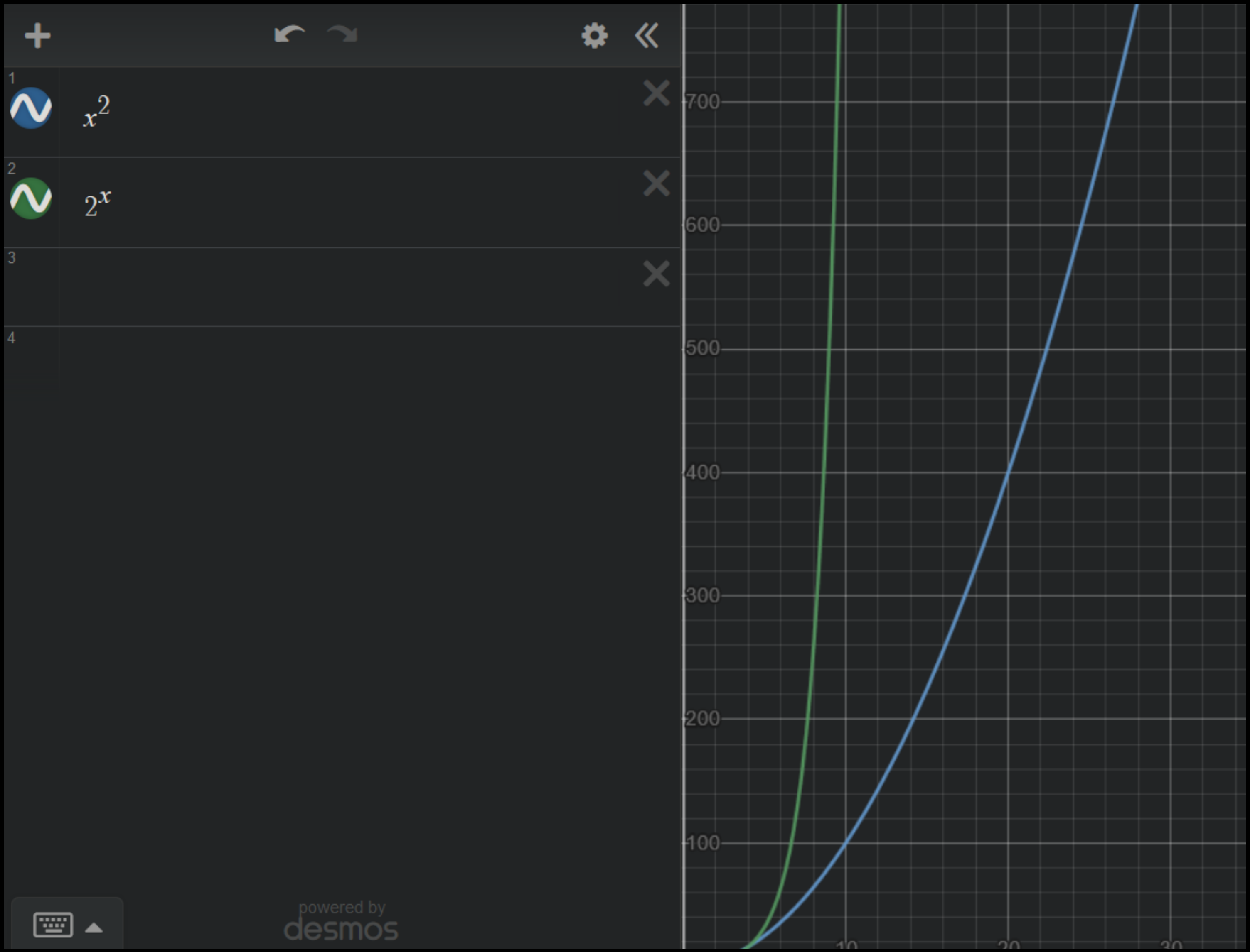
$12! = 1 \times 2 \times \ldots \times 12 = 479{,}001{,}600$

$13! = 1 \times 2 \times \ldots \times 13 = 6{,}227{,}020{,}800$

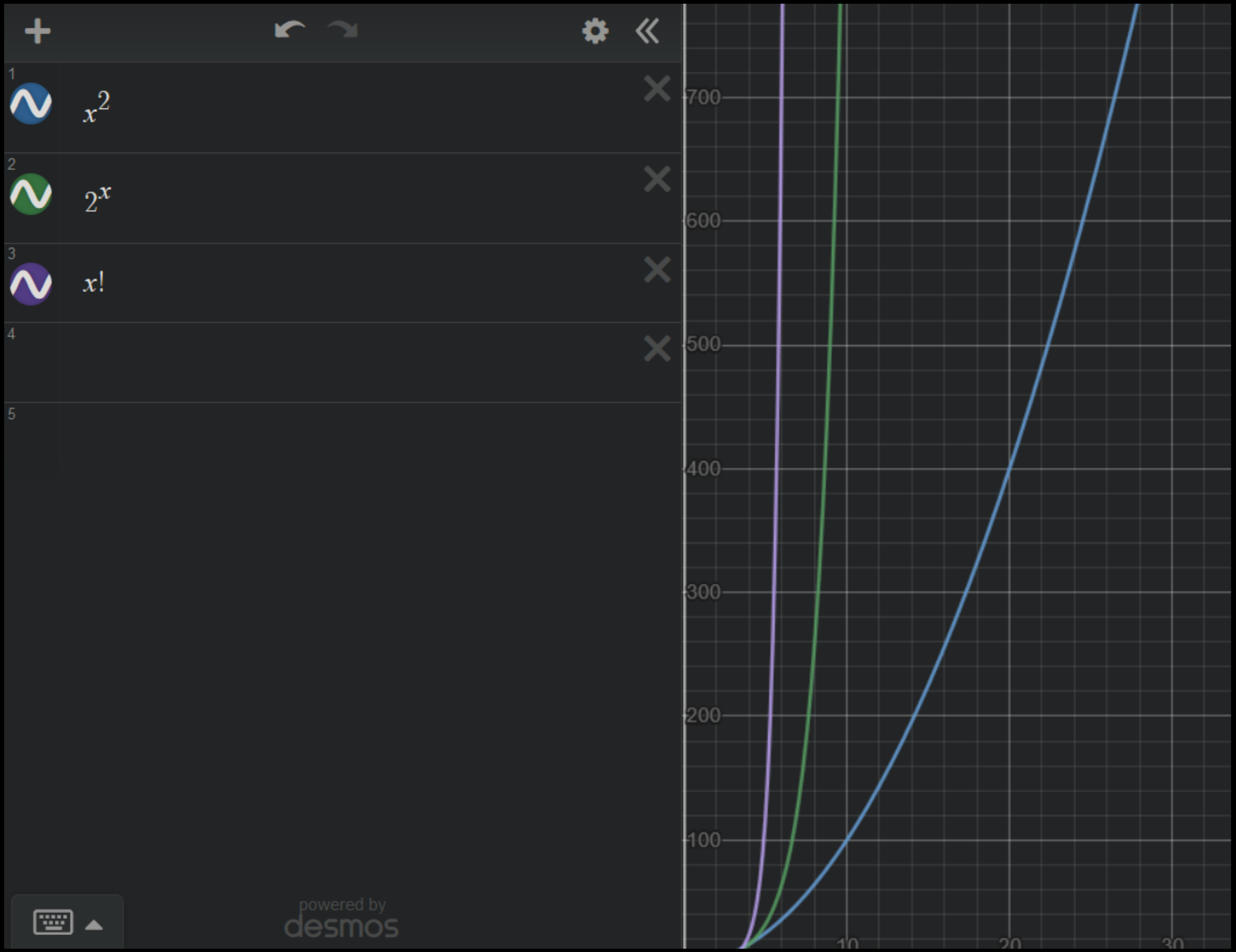
…

$52! = 1 \times 2 \times \ldots \times 52 \approx 8 \times 10^{67}$

Factorials



Factorials



1
x^2
2
2^x
3
$x!$
4
5
powered by
desmos
700
600
500
400
300
200
100
10
20
30

Choosing Stuff

How many ways are there to choose 3 objects, from 5 objects?

I could choose 421 or 245 for example.

There are 10:

123	145
124	234
125	235
134	245
135	345

Choosing Stuff

Here's how we could figure this out systematically:

5 Objects

3 chosen

2 not chosen

1 2 3 4 5

There are 5! ways to arrange the objects.
Divide by 3! and 2! since the order of the objects in the blue and red regions doesn't matter.

$$\text{Total possibilities} = \frac{5!}{3!2!} = 10$$

Choosing Stuff

In General:

n Objects

k chosen — n-k not chosen

1 2 ... k-1 k k+1 ... n-1 n

There are n! ways to arrange the objects.
Divide by k! and (n-k)! since the order of the objects in the blue and red regions doesn't matter.

$$\text{Total possibilities} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Combinations

The previous example of choosing 3 objects from 5 objects would be written:

$$\binom{5}{3}$$

This is said like "5 choose 3".

One interesting thing about combinations is that

$$\binom{5}{3} = \binom{5}{2}$$ or, more generally: $$\binom{n}{k} = \binom{n}{n-k}$$

This is because choosing k objects is the same as not choosing n-k objects.

Think of this as exchanging the red and blue regions in the previous explanation.

Combinations

$\binom{0}{k}$ 1

$\binom{1}{k}$ 1 1

$\binom{2}{k}$ 1 2 1

$\binom{3}{k}$ 1 3 3 1

$\binom{4}{k}$ 1 4 6 4 1

$\binom{5}{k}$ 1 5 10 10 5 1

$\binom{6}{k}$ 1 6 15 20 15 6 1

$\binom{7}{k}$ 1 7 21 35 35 21 7 1

You may notice that any two adjacent numbers add up to the number below them.

This is because:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This triangle is called Pascal's Triangle. Even though it was discovered hundreds of years before Pascal was born.

Week 14 – Jan 19

1. Esteban has a strawberry, a blueberry, a raspberry and a blackberry, and he wants to place them on the corners of a cake. How many ways are there to place the berries if:
    a) He wants to place 1 berry at each corner.
    b) He wants to place 1 or 0 berries at each corner.
    c) He wants to place 1 berry at each corner, but he has 4 berries of each type instead of 1.

2. Jason has a full suit of 13 hearts, if he picks 3 cards, what's the probability he gets:
    a) The Ace b) The King and Queen c) 4, 5, and 6

3. Prove that $\binom{2022}{0} + \binom{2022}{2} + \binom{2022}{4} + \dots + \binom{2022}{2020} + \binom{2022}{2022} = 2^{2021}$

[Hint 1] Use $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (But prove it first!)

[Hint 2] Remember that the number of subsets of a set with n elements is 2^n.