29 September 2021

Problem Solving Club

Introduction to the Club

Introduction

Head of the Club:

Lucas Jacobs

Problem Solving

Number Theory

Prime | Counting Numbers | Composite

1 2 3 4 5 6 7 8 9 10 11 12

$4 \rightarrow 2^2$, $6 \rightarrow 2 \times 3$, $8 \rightarrow 2^3$, $9 \rightarrow 3^2$, $10 \rightarrow 2 \times 5$, $12 \rightarrow 2^2 \times 3$

Includes The Most Important Unsolved Problem in Mathematics: The Riemann Hypothesis

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Number Theory

Prime   Counting Numbers   Composite

1 2 3 4 5 6 7 8 9 10 11 12

2^2 2×3 2^3 3^2 2×5 $2^2 \times 3$

Sample Problems:

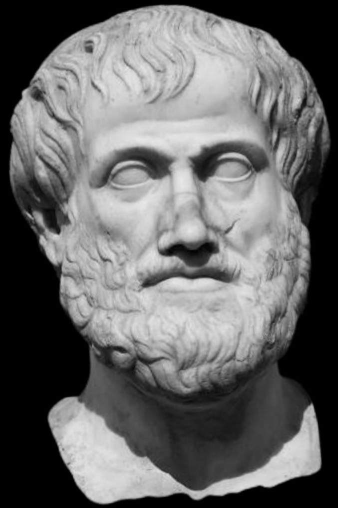
- Finding all numbers that satisfy some property

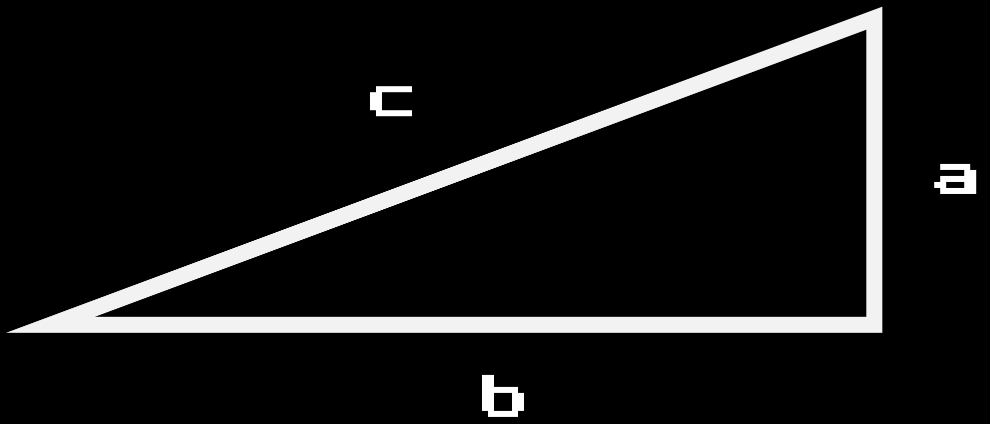
Ex: Find all integers n where $6(3/2)^n$ is an integer

- Finding properties of the divisors of a number

Ex: Find the number of factors of 14!

Geometry

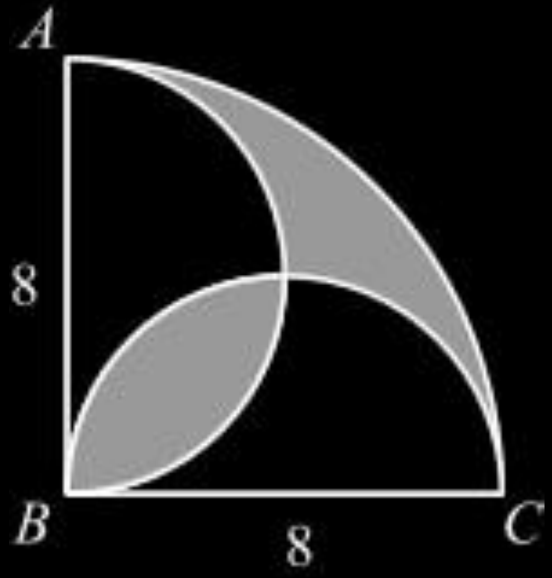




$$a^2 + b^2 = c^2$$

Sample Problem:

What is the area of the shaded region?

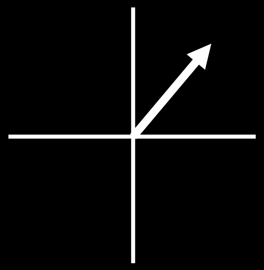


Algebra

$$6x + 20 = 0$$

$$x^2 + 3x - 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

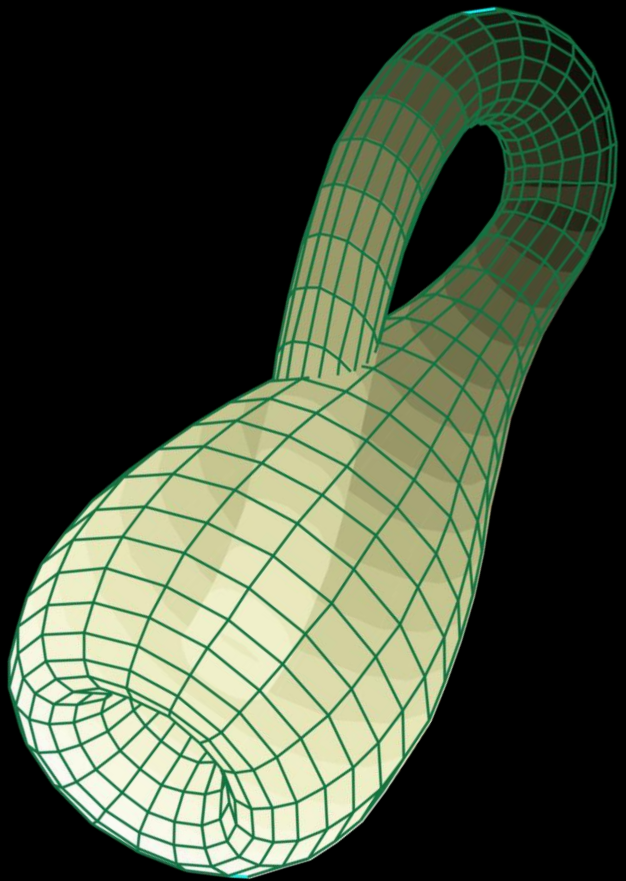
Sample Problems:

- Finding the maximum/minimum value of some function
- Finding the formula for some number sequence
- Proving an inequality
- Finding the zeros of a polynomial

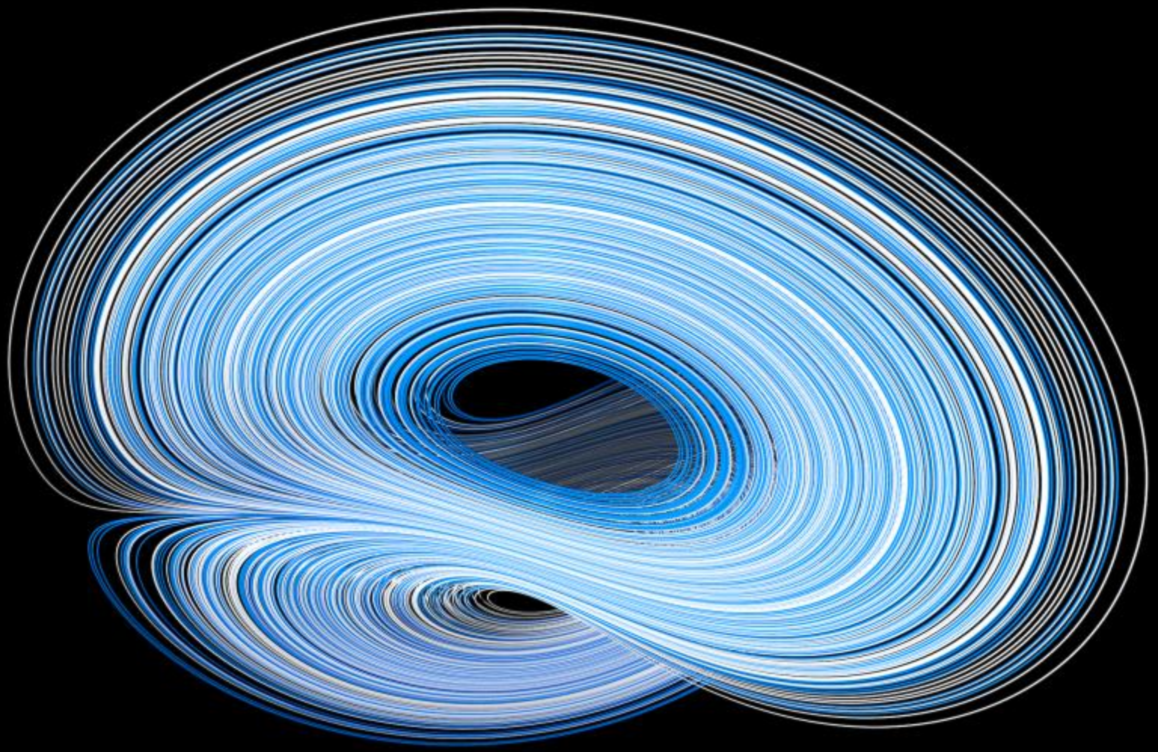
Example:
Can you find the number of ways to tile a 2×8 rectangle with 1×2 blocks?

Other Areas of Math

Topology



Analysis



How it Feels to Solve Problems

Aha Moments

How it Feels to Solve Problems

$7x^2 - 294x + 2639 = 0$

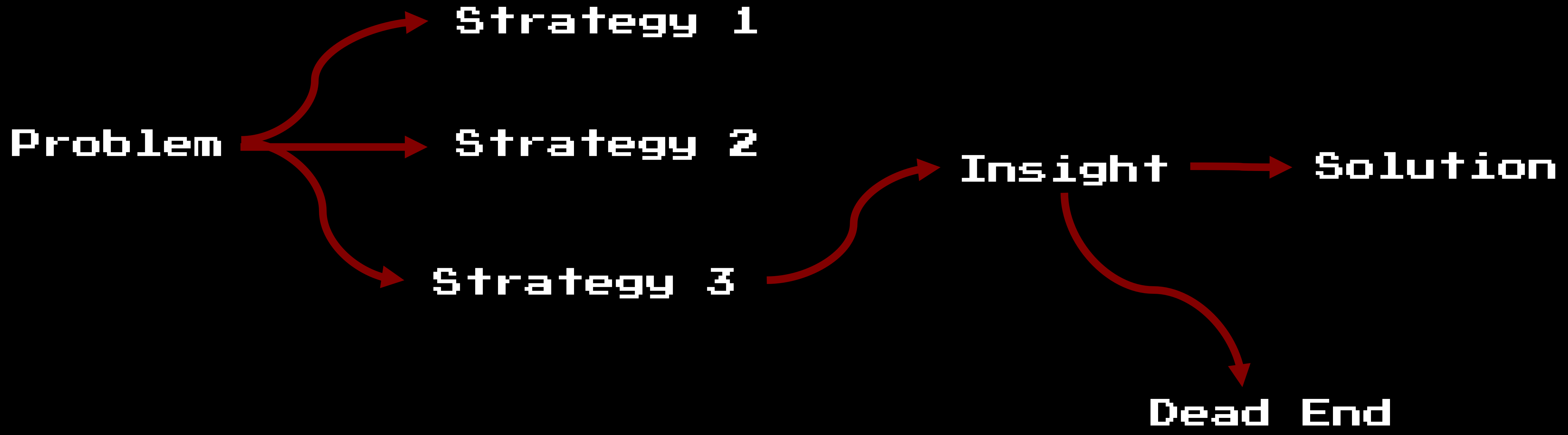
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-294) \pm \sqrt{(-294)^2 - 4(7)(2639)}}{2(7)}$$

$x = 12,\ 29$

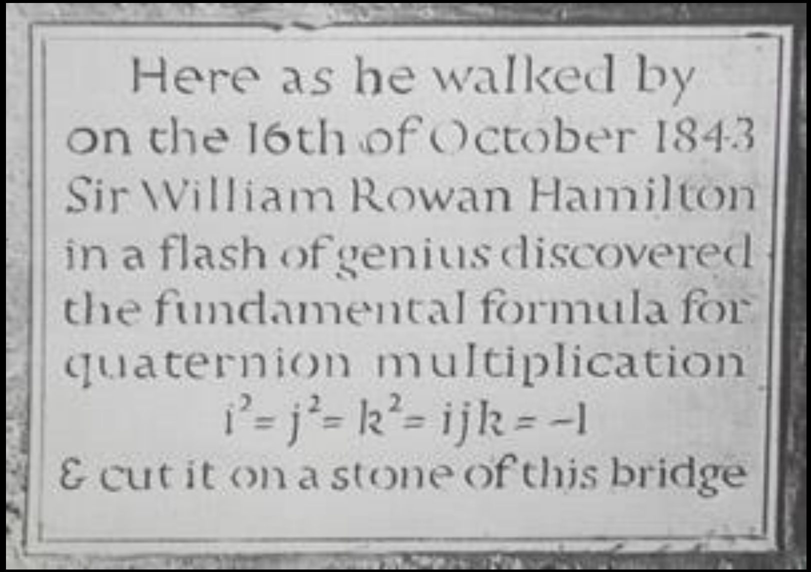
How it Feels to Solve Problems

$7x^2 - 294x + 2639 = 0 \longrightarrow x = 12, 29$

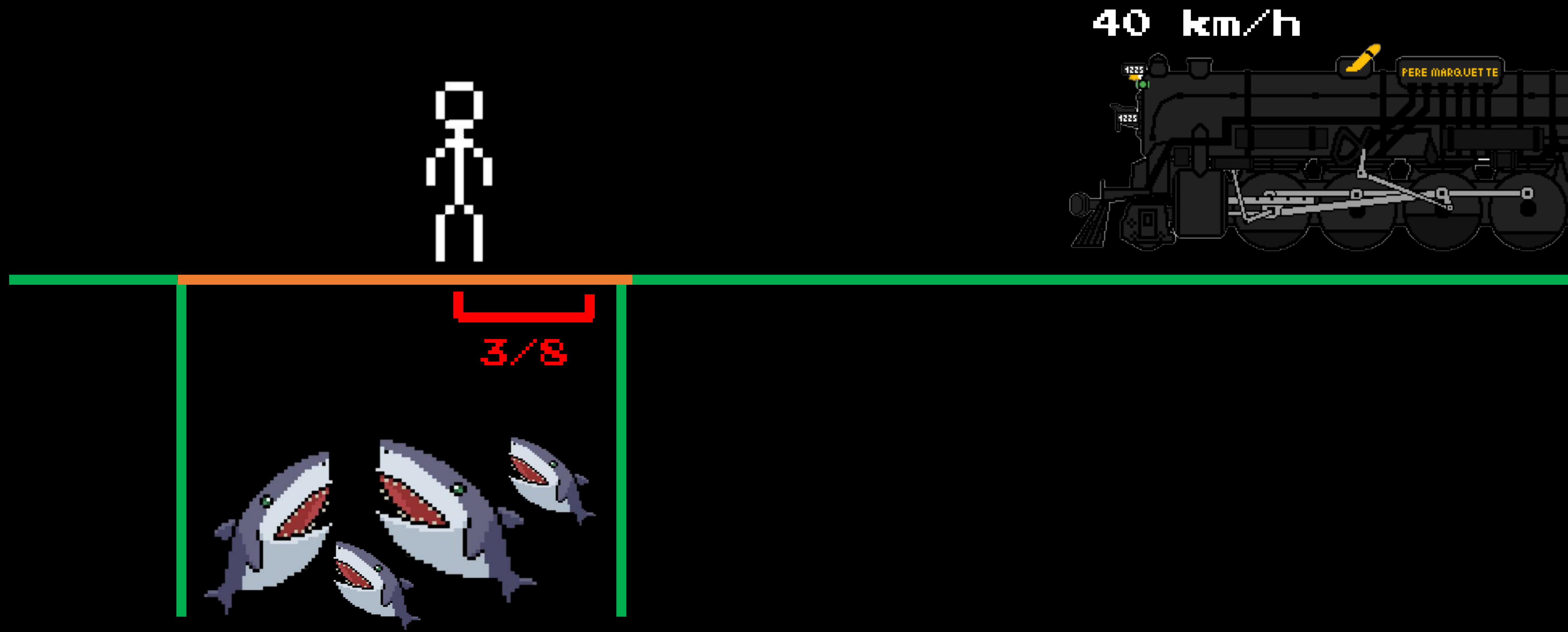


How it Feels to Solve Problems

Take Breaks



The Train Problem



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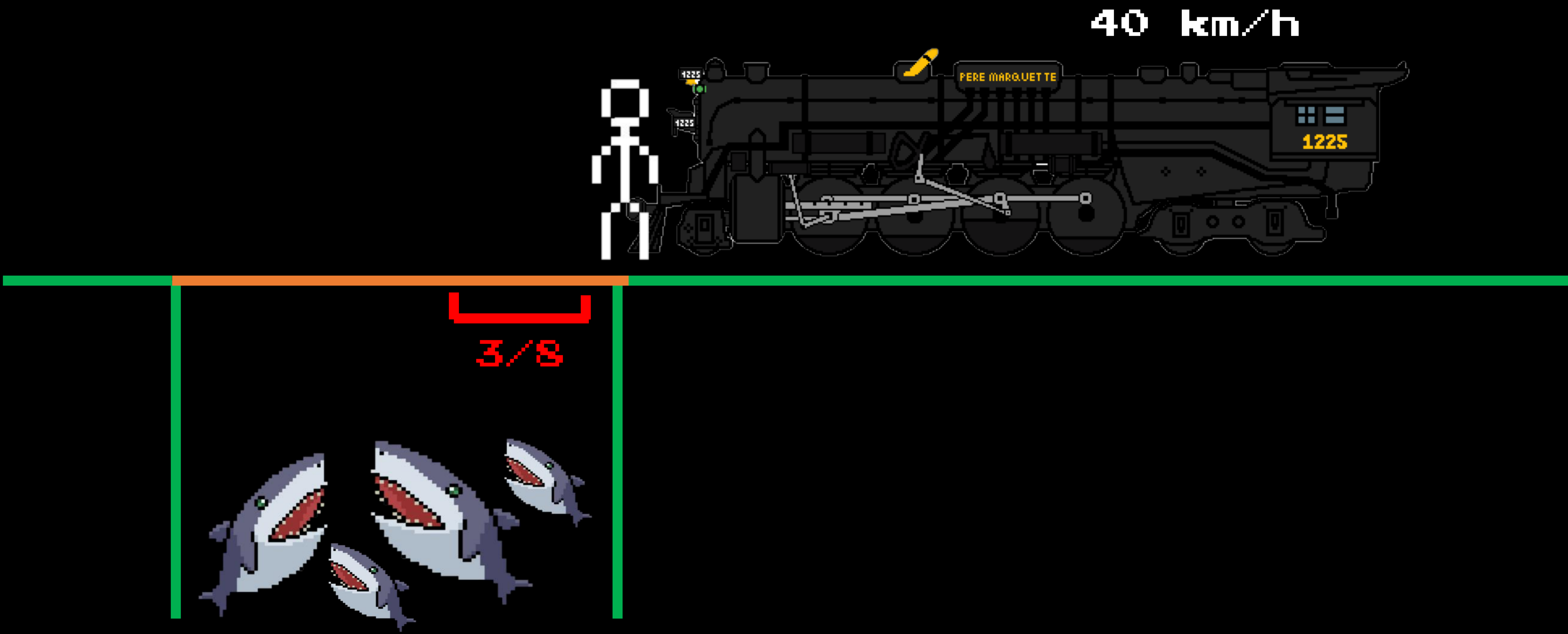

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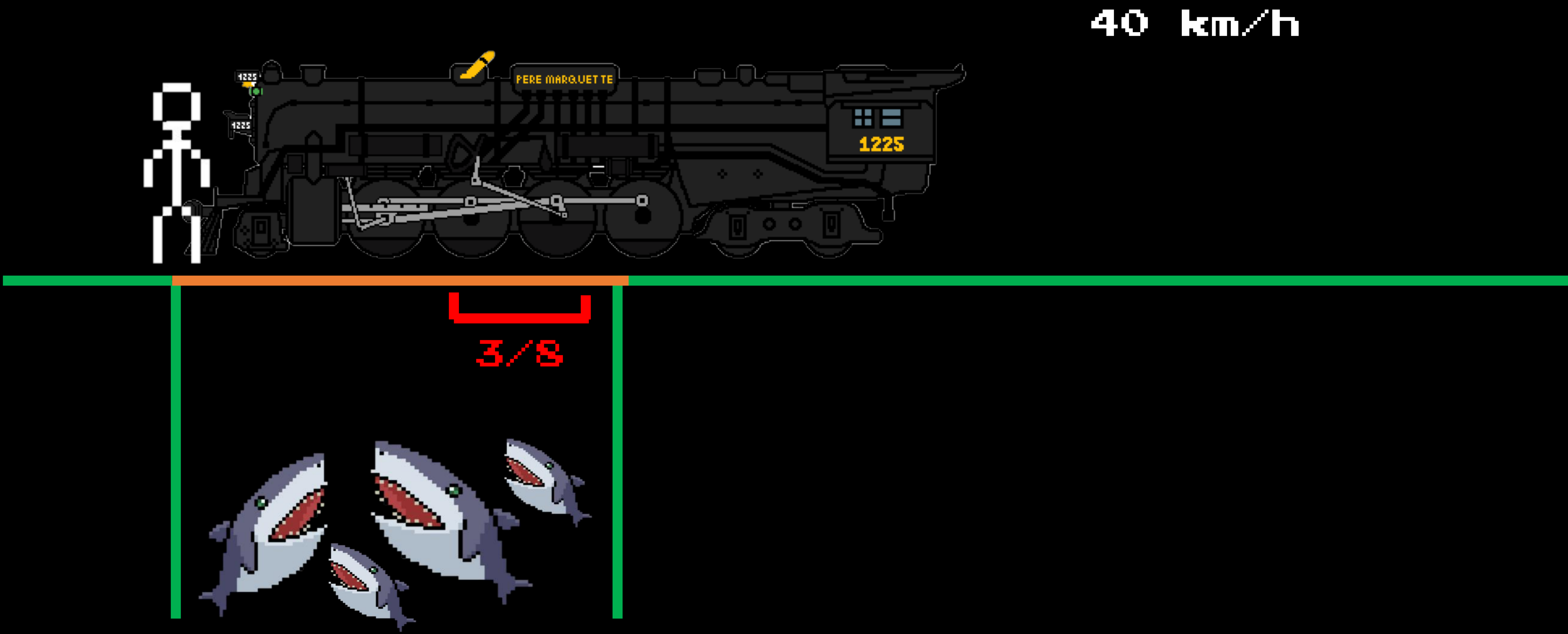

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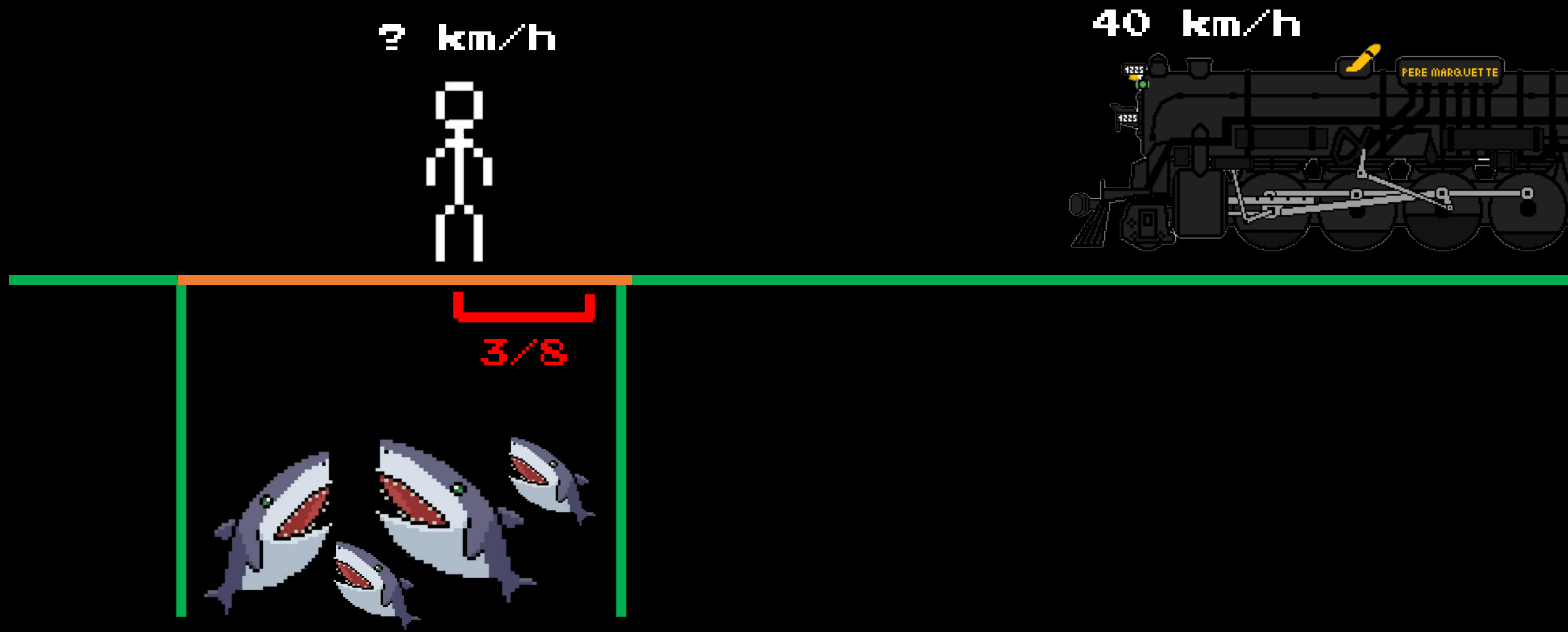


What's the Point?

Math Contests:

- COMC (All Grades) – 28 Oct 21 – $25
- CIMC/CSMC (All Grades) – 17 Nov 21 – $12
- CCC (All Grades) – 16 Feb 22 – $8
- Pascal/Cayley/Fermat (9/10/11) – 23 Feb 22 – $5
- Euclid (12) – 5 Apr 22 – $16
- CTMC (All Grades) – 7 Apr 22
- Fryer/Galois/Hypatia (9/10/11) – 12 Apr 22 – $10

What's the Point?

"Teaching to solve problems is education of the will. Solving problems which are not too easy …, the student learns to persevere through unsuccess, to appreciate small advances, to wait for the essential idea, to concentrate with all their might when it appears. If the student had no opportunity in school to familiarize themselves with the varying emotions of the struggle for the solution, their mathematical education failed in the most vital point."

—George Polya

Proofs

An argument that would convince a reasonable person that your solution is correct.

How many factors does 14! have?

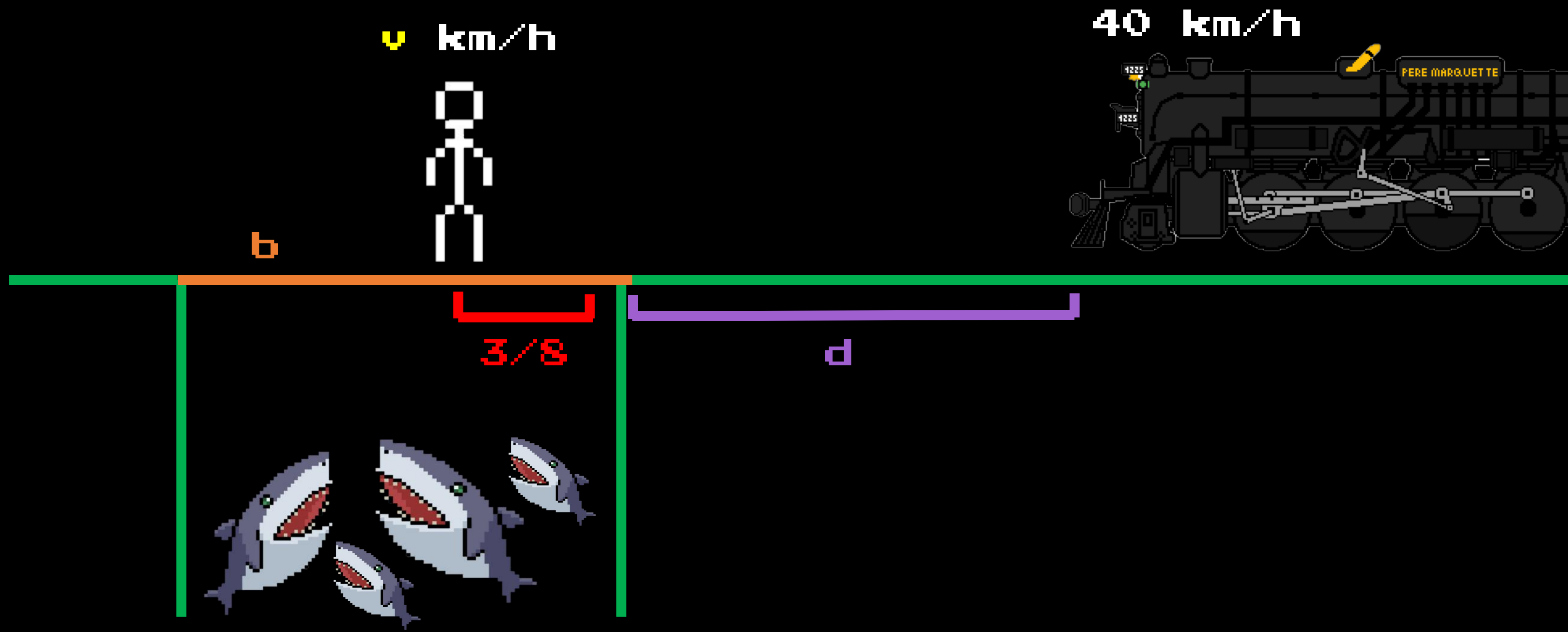
? ? ?

? 2592 ?

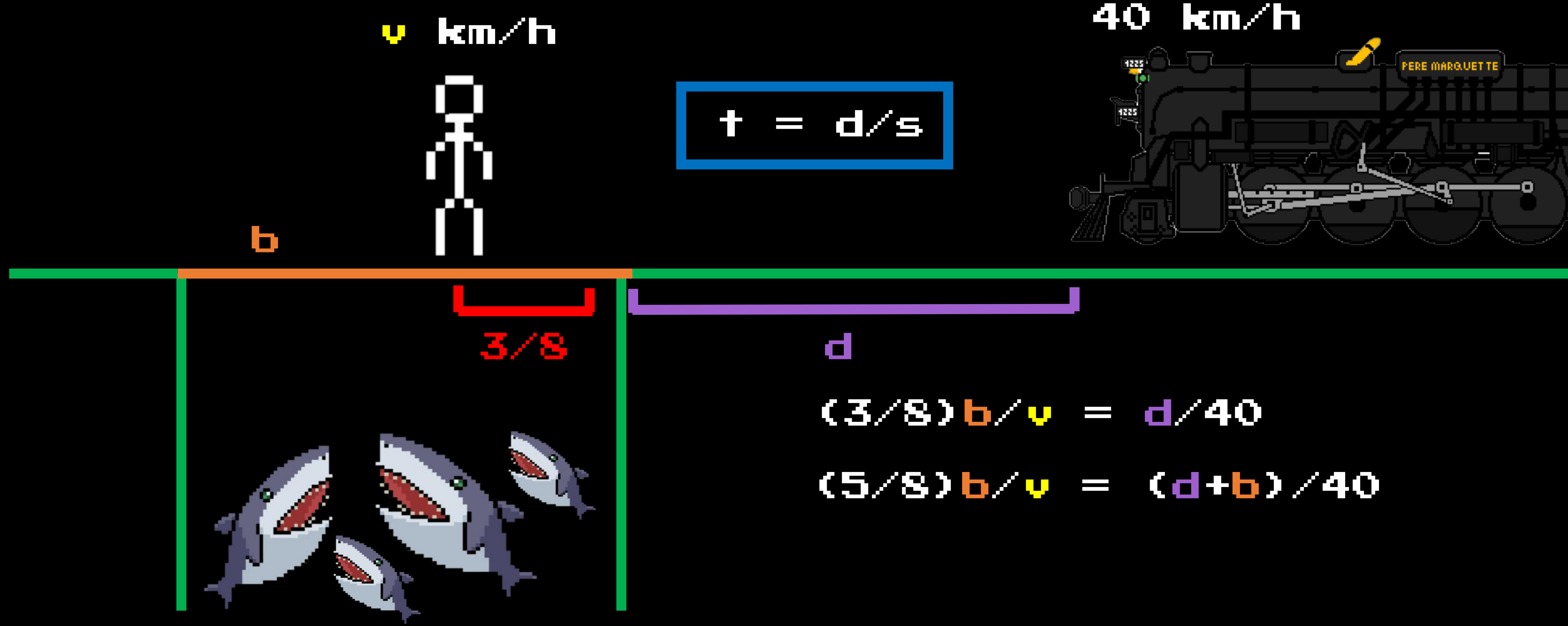
? ? ? ? ?

Proofs are important

Return of The Train



Return of The Train



$$t = d/s$$

$$(3/8)b/v = d/40$$

$$(5/8)b/v = (d+b)/40$$

Return of the Train

(1) $(3/8)b/v = d/40$

(2) $(5/8)b/v = (d+b)/40$

(1) $(3/8)40/v = d/b$

$15/v = d/b$

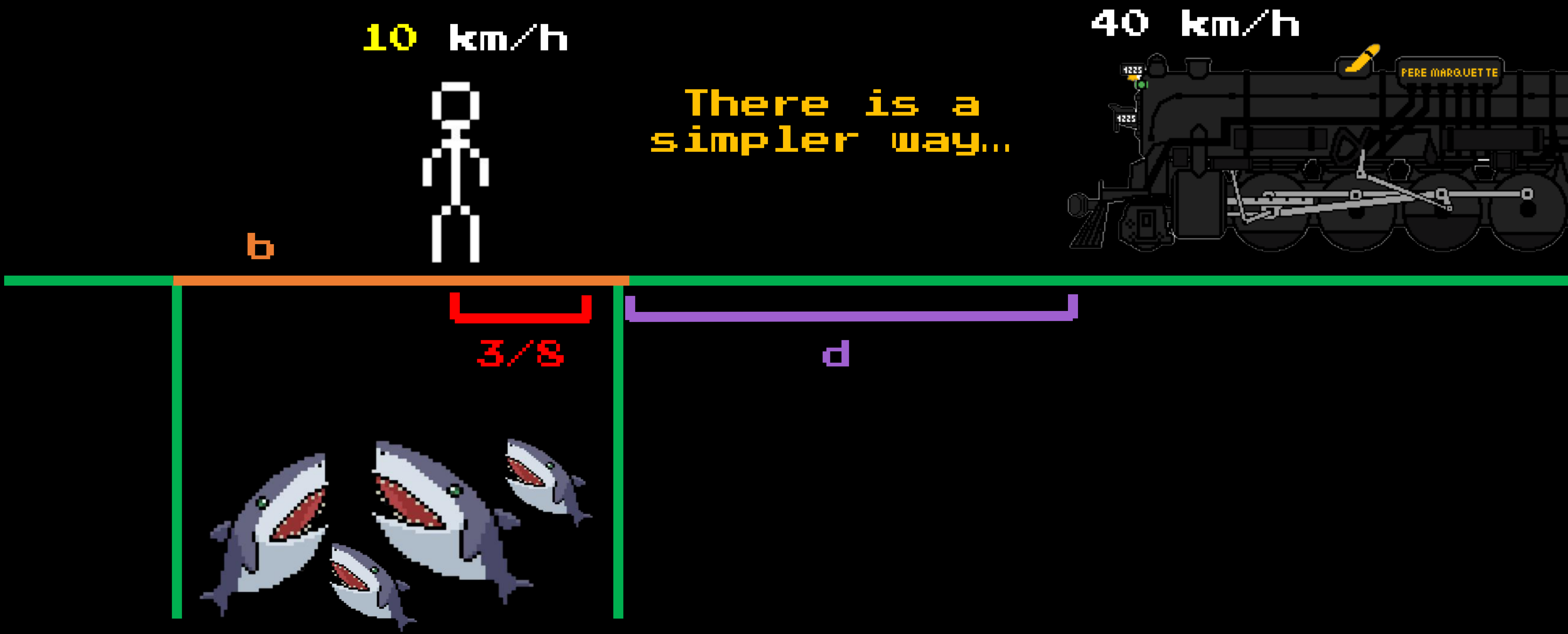
(2) $(5/8)40/v = (d+b)/b$

$25/v = d/b + 1$

$25/v - 1 = d/b$

$15/v = 25/v - 1 \rightarrow v = 10.$

Return of The Train



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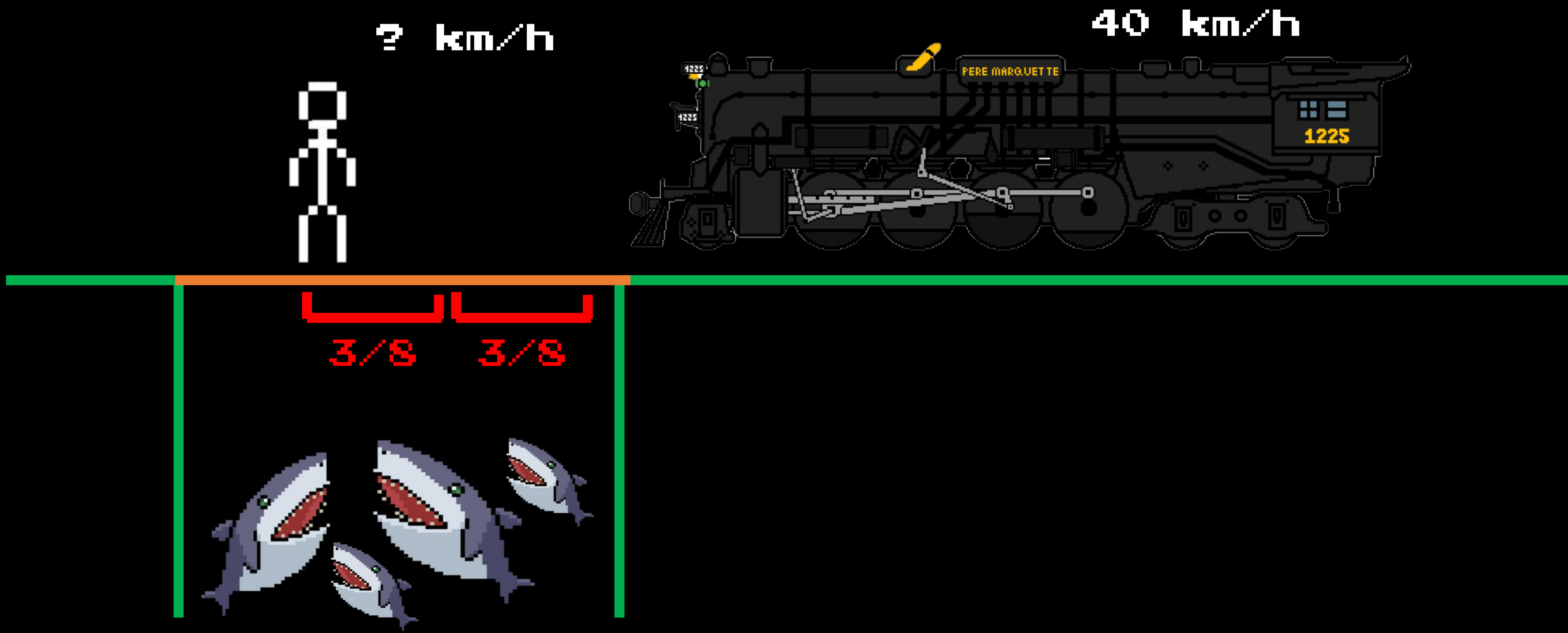

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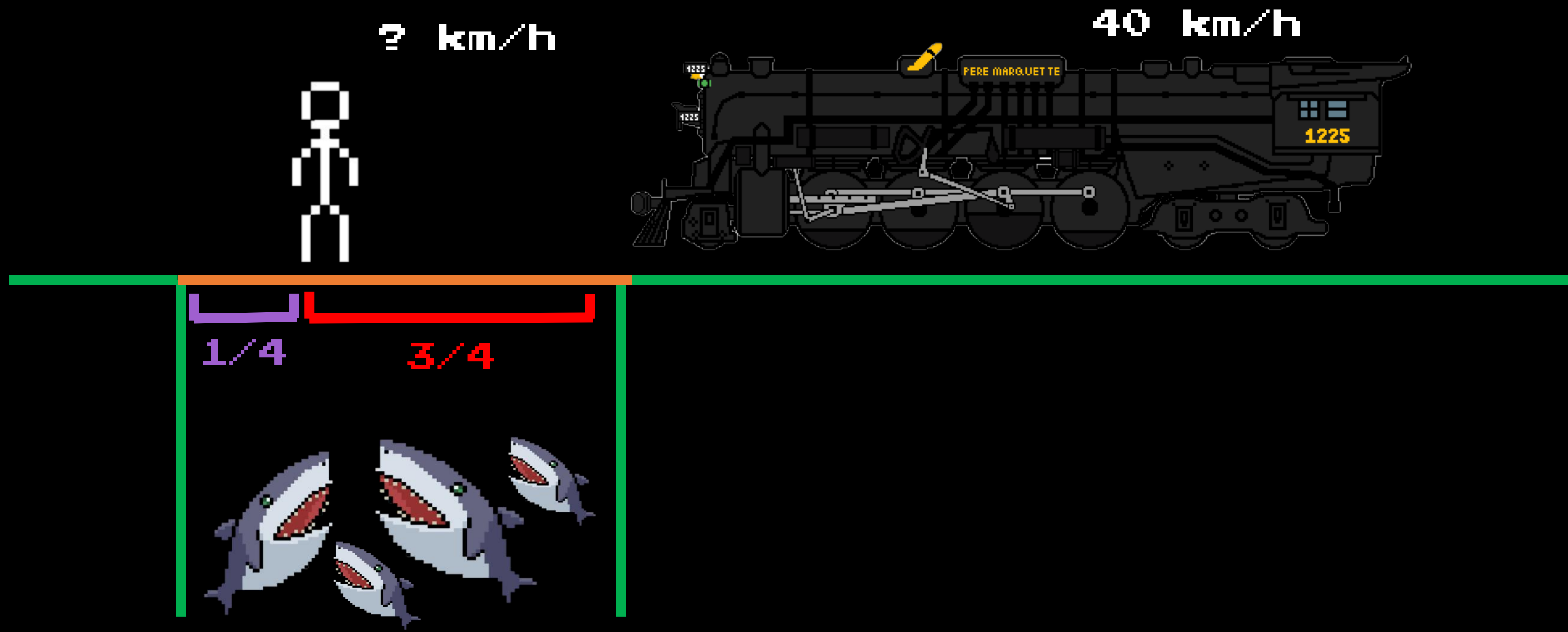

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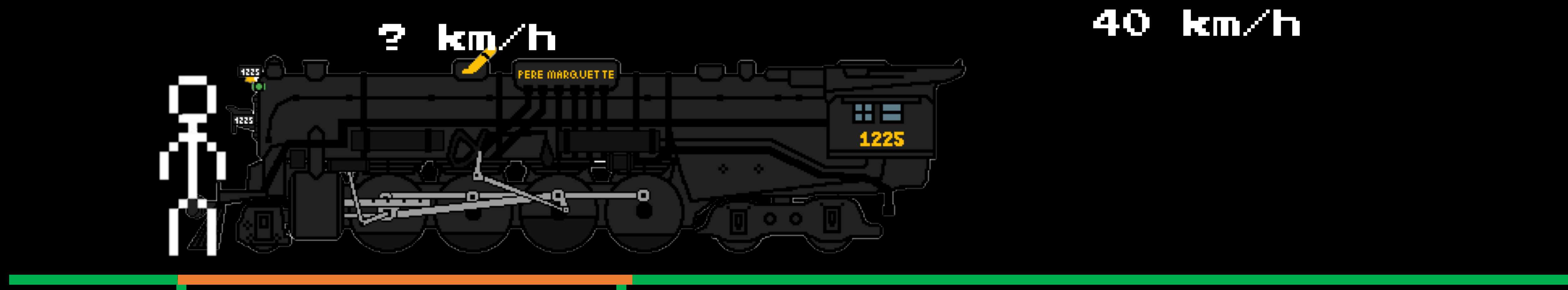

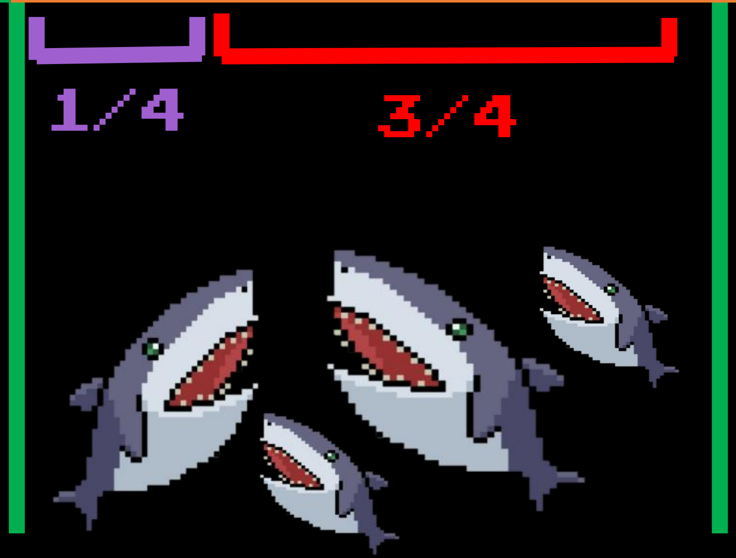

∴ We must travel 4 times as slow as the train.

Our speed = 40/4 = 10 km/h

Train Problem: Full Proof

Since you can run 3/8 of the bridge in the time it takes the train to reach the start of the bridge, if you decide to run 3/8 of the way to the end of the bridge, you will be 3/8 + 3/8 = 3/4 of the way across the bridge when the train meets the start of the bridge.

Your distance to the end of the bridge is now 1/4 the length of the bridge, while train's distance is the full length of the bridge. Since you arrive there at the same time as the train, you must run 1/4 the speed of the train. The train travels at 40 km/h, so you must run at 10 km/h. ∎

Week 1 – Sept 29

Write a proof for each of the following problems by next week.

1. Three pumpkins are weighed 2 at a time in all possible ways. The weights are 12 kg, 13 kg, and 15 kg. What is the mass of the lightest pumpkin?

2. Is $n^2 - n + 41$ prime for every positive integer n?

3. What is the units digit of 3^{2021}?